(Volumes of) hyperbolic manifolds

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(joint, over the years, with John Ratcliffe, Steve Tschantz and Bob Howlett)

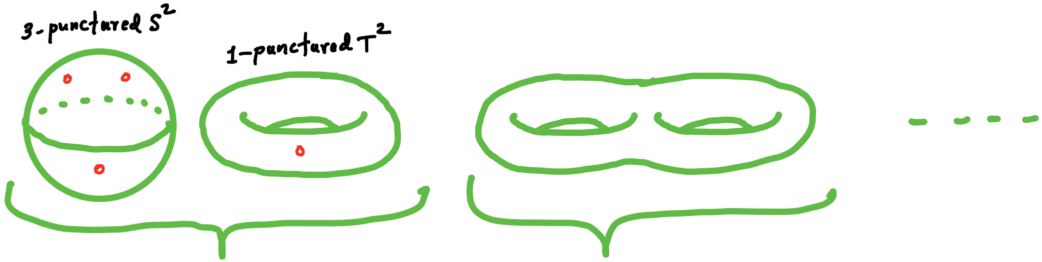
1 Hyperbolic manifolds:

- X = complete Riemannian mfld. constant sectional curvature -1
- $\mathcal{H}(n) = \{$n-dim. orientable hyp. mflds./isometry$\}$

  $\mathrm{vol}: \mathcal{H}(n) \to \mathbb{R}^{>0} \cup \{\infty\}$
- image vol well-ordered (discrete for $n \neq 3$)

Image (n even): $\mathrm{vol}(X) = \kappa_n \chi(X)$

$n=2$:



$|\chi|=1$ $|\chi|=2$ $|\chi|=3$ ...

$n=4$: cusped 24-cell/~ (non-compact) (smallest compact unknown)

$n=6$: see later! (non-compact) (ditto)

② Algebraic construction:

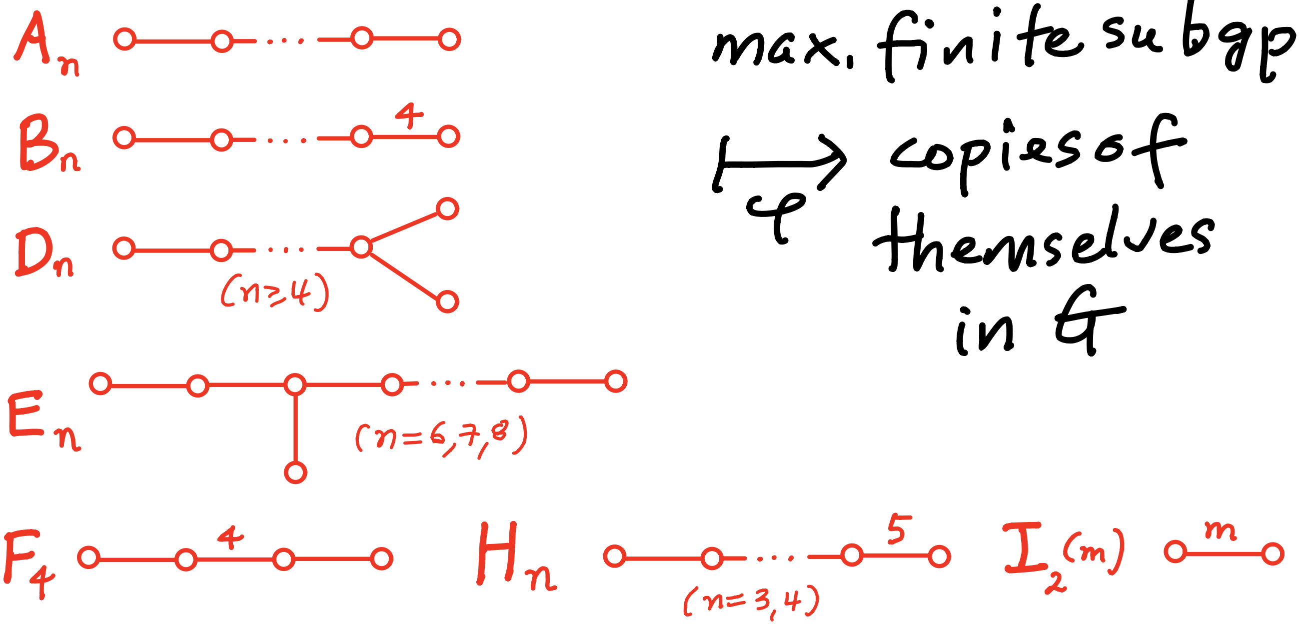
- $X = \mathbb{H}^n/\Gamma$
  - → Γ acting freely prop. disc. by isometries
  - → Γ torsion-free discrete $\leq \mathrm{Isom}\,\mathbb{H}^n$

(rigidity$_{n\geq 3}$: $\mathbb{H}^n/\Gamma_1 \cong \mathbb{H}^n/\Gamma_2 \Leftrightarrow \Gamma_1 \cong \Gamma_2$)

- $\Gamma \leq W$ a reflection (or Coxeter) gp. $\curvearrowright \mathbb{H}^n$

(has a symbol)

- maximal finite subsymbols from list:



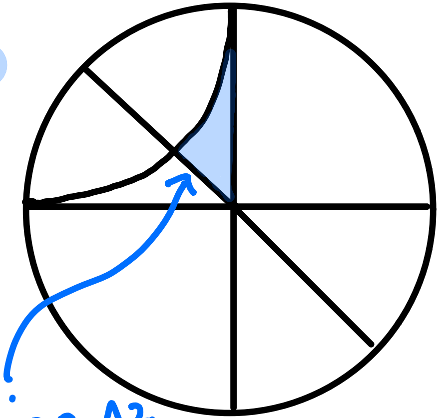
- $\varphi : W \to G$ (finite)

$\ker \varphi$ torsion-free

$\Longleftrightarrow$

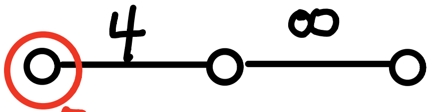
max. finite subgps $\overset{\varphi}{\longmapsto}$ copies of themselves in G

③ Toy example (all $\llcorner$ angled technology)

- 

region Δ^2

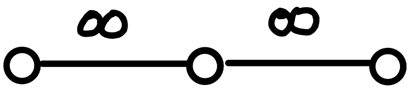
$W_0 \leqslant \mathrm{Isom}\,\mathbb{H}^2$



subgroup Σ

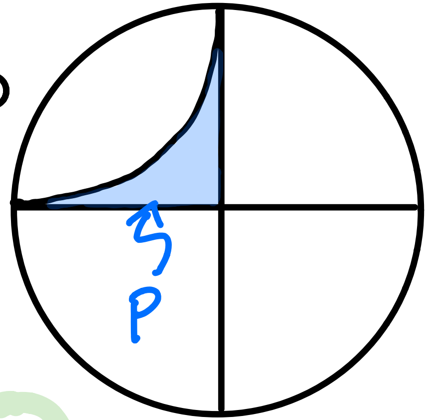
- $P = \Sigma\Delta^2$

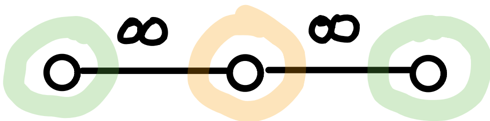
all right angled



symbol

This is W



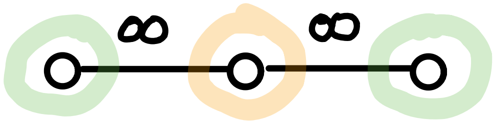


maximal finite subgroups

- $\varphi : W \to \mathbb{Z}_2 \times \mathbb{Z}_2$ ($\approx$ 2-dim. $\mathbb{F}_2$-space V)

  generators $\longmapsto$ anything!

- kernel torsion-free (old trick):

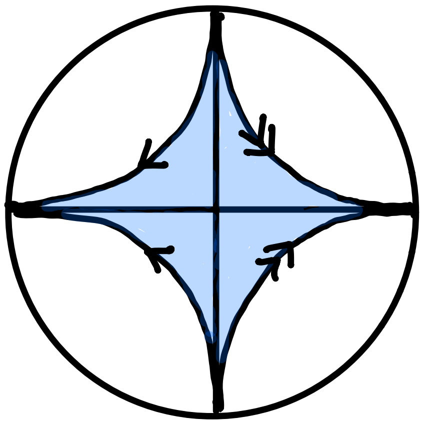


independent vectors in V

Eg: $\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$

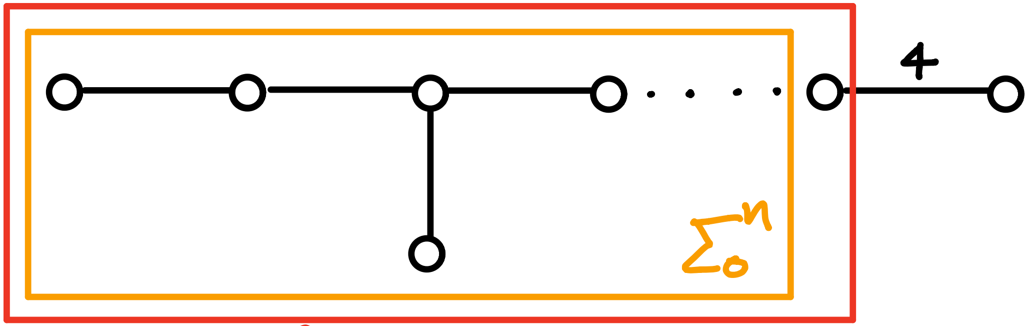
$\mathbb{H}^2/\Gamma \cong S^2 \setminus 3$ pts.

$|\chi| = 1$



④ In dimensions $4 \leq n \leq 8$

- $W_0 =$



$\Sigma^n = A_4, D_5, E_6, E_7, E_8$

$\leq \mathrm{Isom}\,\mathbb{H}^n$

- $P = \Sigma^n \Delta^n$ all right-angled polytope

This is W

- $W \to \mathbb{Z}_2^n$ label generators of W by vectors in $\mathbb{F}_2^n = V$
- $V \equiv$ Tits representation of Σ^n
  $L \subset V$ root lattice
- choose $v \in V$ s.t. $v \perp \Sigma_0^n$; then
  Σ^n-orbit of $v \xleftrightarrow{1-1}$ faces of P
  Now reduce L mod 2!
- $n=4: \chi=1$ $n=6: \chi=-8$