

Aspects of Defects and Boundaries in Integrable Field Theory

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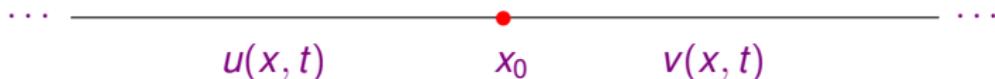
September 2016

Contents

- Boundaries and defects (eg impurities, shocks, dislocations) are ubiquitous in nature
- What are their properties within an integrable field theory in two-dimensional space-time?
- The purpose here is to consider how they may be related
 - Examples of integrable defects and the special role played by energy-momentum conservation and Bäcklund transformations
 - Solitons scattering with defects and some curious effects
 - Defects in integrable quantum field theory and transmission matrices
 - Boundaries revisited
- With: P Bowcock (Durham-Maths), C Zambon (Durham-Physics)]

An integrable discontinuity - Bowcock, EC, Zambon (2003)

Start with a single selected point on the x -axis, say $x_0 = 0$, and denote the field to the left ($x < 0$) by u , and to the right ($x > 0$) by v :



Field equations in separated domains:

$$\partial^2 u = -\frac{\partial U}{\partial u}, \quad x < 0, \quad \partial^2 v = -\frac{\partial V}{\partial v}, \quad x > 0$$

• U, V both integrable (but not linear), how can the fields u, v be 'sewn' together at x_0 to preserve integrability?

One natural choice (δ -impurity) would be to put

$$u(0, t) = v(0, t), \quad u_x(0, t) - v_x(0, t) = F(u(0, t)),$$

- integrability is lost - Goodman, Holmes, Weinstein (2002).

- Problem: there is a distinguished point, translation symmetry is lost and conservation laws - at least some of them - (for example, momentum), will be violated unless the impurity contributes compensating terms.

Consider the field contributions to energy-momentum:

$$P^\mu = \int_{-\infty}^0 dx T^{0\mu}(u) + \int_0^{\infty} dx T^{0\mu}(v).$$

Using the field equations, can we arrange

$$\frac{dP^\mu}{dt} = - [T^{1\mu}(u)]_{x=0} + [T^{1\mu}(v)]_{x=0} = - \frac{dD^\mu(u, v)}{dt}$$

with the right hand side depending only on the fields at $x = 0$?

If so, $P^\mu + D^\mu$ is conserved with D^μ being the defect contribution.

- Only a few possible sewing conditions (and bulk potentials U, V) are permitted for this to work.

Consider the field contribution to energy and calculate

$$\frac{dP^0}{dt} = [u_x u_t]_0 - [v_x v_t]_0.$$

Setting

$$u_x = v_t + X(u, v), \quad v_x = u_t + Y(u, v)$$

we find

$$\frac{dP^0}{dt} = u_t X - v_t Y.$$

This is a total time derivative if

$$X = -\frac{\partial D^0}{\partial u}, \quad Y = \frac{\partial D^0}{\partial v},$$

for some D^0 . Then

$$\frac{dP^0}{dt} = -\frac{dD^0}{dt}.$$

- Expected anyway since time translation remains good.

On the other hand, for momentum

$$\begin{aligned}\frac{dP^1}{dt} &= - \left[\frac{u_t^2 + u_x^2}{2} - U(u) \right]_{x=0} + \left[\frac{v_t^2 + v_x^2}{2} - V(v) \right]_{x=0} \\ &= \left[-v_t X + u_t Y - \frac{X^2 - Y^2}{2} + U - V \right]_{x=0} = - \frac{dD^1(u, v)}{dt}\end{aligned}$$

This is a total time derivative provided the first piece is a perfect differential and the second piece vanishes. Thus

$$X = - \frac{\partial D^0}{\partial u} = \frac{\partial D^1}{\partial v}, \quad Y = \frac{\partial D^0}{\partial v} = - \frac{\partial D^1}{\partial u},$$

In other words

$$\frac{\partial^2 D^0}{\partial v^2} = \frac{\partial^2 D^0}{\partial u^2}, \quad \frac{1}{2} \left(\frac{\partial D^0}{\partial u} \right)^2 - \frac{1}{2} \left(\frac{\partial D^0}{\partial v} \right)^2 = U(u) - V(v).$$

Highly constraining - just a few possible combinations U, V, D^0 ?

- sine-Gordon, Liouville, massless free, or, massive free.

For example, if $U(u) = m^2 u^2 / 2$, $V(v) = m^2 v^2 / 2$, D^0 turns out to be

$$D^0(u, v) = \frac{m\sigma}{4}(u+v)^2 + \frac{m}{4\sigma}(u-v)^2,$$

and σ is a free parameter.

- Note: the Tzitzéica (aka BD, MZS, $a_2^{(2)}$ affine Toda) potential

$$U(u) = e^u + 2e^{-u/2}$$

is **not** possible.

- There is a Lagrangian description of this type of defect (type I):

$$\mathcal{L} = \theta(-x)\mathcal{L}(u) + \delta(x) \left(\frac{uv_t - u_t v}{2} - D^0(u, v) \right) + \theta(x)\mathcal{L}(v).$$

sine-Gordon - Bowcock, EC, Zambon (2003, 2004, 2005)

Choosing u, v to be sine-Gordon fields (and scaling the coupling and mass parameters to unity), the allowed possibilities are:

$$D^0(u, v) = -2 \left(\sigma \cos \frac{u+v}{2} + \sigma^{-1} \cos \frac{u-v}{2} \right),$$

where σ is a constant, to find

$$x < 0 : \quad \partial^2 u = -\sin u,$$

$$x > 0 : \quad \partial^2 v = -\sin v,$$

$$x = 0 : \quad u_x = v_t - \sigma \sin \frac{u+v}{2} - \sigma^{-1} \sin \frac{u-v}{2},$$

$$x = 0 : \quad v_x = u_t + \sigma \sin \frac{u+v}{2} - \sigma^{-1} \sin \frac{u-v}{2}.$$

- The final two are a Bäcklund transformation **frozen** at $x = 0$.
- The defect could be anywhere - essentially topological
- Higher conserved charges and other properties have been checked.

Solitons and defects - Bowcock, EC, Zambon (2005)

The sine-Gordon model allows solitons and antisolitons.

Consider a soliton incident from $x < 0$.

It will not be possible to satisfy the sewing conditions (in general, for all times) unless a similar soliton emerges into the region $x > 0$:

$$x < 0 : \quad e^{iu/2} = \frac{1 + iE}{1 - iE},$$

$$x > 0 : \quad e^{iv/2} = \frac{1 + izE}{1 - izE},$$

$$E = e^{ax+bt+c}, \quad a = \cosh \theta, \quad b = -\sinh \theta,$$

where z is to be determined. It is also useful to set $\sigma = e^{-\eta}$.

- We find....

$$z = \coth\left(\frac{\eta - \theta}{2}\right) \quad \theta > 0$$

Remarks:

- $\eta < \theta$ implies $z < 0$; ie the soliton emerges as an anti-soliton.
 - the final state will contain a discontinuity of magnitude 4π at $x = 0$.
- $\eta = \theta$ implies $z = \infty$ and there is **no** emerging soliton.
 - the energy-momentum of the soliton is captured by the 'defect'.
 - the topological charge is also captured by a discontinuity 2π .
- $\eta > \theta$ implies $z > 0$; ie the soliton retains its character.

Comments

- Defects at $x = x_1 < x_2 < x_3 < \dots < x_n$ behave independently
 - each contributes a factor z_i for a total $z = z_1 z_2 \dots z_n$.
- Each component of a multisoliton is affected separately
 - thus at most one can be 'filtered out'.
- Since a soliton can be absorbed, can a starting configuration with $u = 0, v = 2\pi$ decay into a soliton?
 - needs quantum mechanics to provide the probability for decay.
- Contrast **Estabrook - Wahlquist (1973)**
 - a Bäcklund transformation 'creates' a soliton.
- Defects can also move (with constant speed), and scatter.

- Note:

In the free case, with a wave incident from the left half-line

$$u = (e^{ikx} + Re^{-ikx}) e^{-i\omega t}, \quad v = T e^{ikx} e^{-i\omega t}, \quad \omega^2 = k^2 + m^2,$$

we find:

$$R = 0, \quad T = -\frac{(i\omega - m \sinh \eta)}{(ik + m \cosh \eta)} = -i \frac{\sinh\left(\frac{\theta - \eta}{2} - \frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta - \eta}{2} + \frac{i\pi}{4}\right)}, \quad \sigma = e^{-\eta}$$

- Unusual to have no reflection from an impurity.
- No bound state (for η real).
- In contrast, for δ -impurity:

$$u(0, t) = v(0, t), \quad u_x(0, t) - v_x(0, t) = \lambda u(0, t),$$

$$R = -\frac{\lambda}{\lambda + 2ik}, \quad T = \frac{2ik}{\lambda + 2ik}$$

Bound state at $k = i\lambda/2$ if $2m > \lambda > 0$.

Generalisations

- What about Tzitzéica ($a_2^{(2)}$ affine Toda)?
- Multi-component fields - what about other affine Toda field theories?
 - only the $a_n^{(1)}$ affine Toda theories can work - EC, Zambon (2009)
 - Bäcklund transformations are similar - Fordy, Gibbons (1980)
- What about nonlinear Schrödinger, KdV, mKdV, etc, etc?
 - yes, see Caudrelier, Mintchev, Ragoucy (2004,) EC, Zambon (2005), Caudrelier (2008), . . .
- Is the setup genuinely integrable? For an alternative (algebraic) approach see Avan, Doikou (2012, 2013); Doikou (2014, 2016)
- What about SUSY? See, for example, Gomes, Ymai, Zimerman (2008); Aguirre, Gomes, Spano, Zimerman (2015)
- How do the sewing conditions work for finite gap solutions of sine-Gordon (or KdV, NLS, etc)? EC, Parini (in preparation)

Classical type II defect - EC, Zambon (2009)

Consider two relativistic field theories with fields u and v , and add a new degree of freedom $\lambda(t)$ at the defect location:

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x) ((u - v)\lambda_t - D^0(\lambda, u, v))$$

Then the usual Euler-Lagrange equations lead to

- equations of motion:

$$\partial^2 u = -\frac{\partial U}{\partial u} \quad x < 0, \quad \partial^2 v = -\frac{\partial V}{\partial v} \quad x > 0$$

- defect conditions at $x = 0$

$$u_x = \lambda_t - D_u^0 \quad v_x = \lambda_t + D_v^0 \quad (u - v)_t = -D_\lambda^0.$$

As before, consider momentum

$$P^1 = - \int_{-\infty}^0 dx u_t u_x - \int_0^{\infty} dx v_t v_x,$$

and seek a functional $D^1(u, v, \lambda)$ such that $P_t^1 \equiv -D_t^1$.

As before, implies constraints on U, V, D^1 .

Putting $q = (u - v)/2$, $p = (u + v)/2$ these are:

$$D_p^0 = -D_\lambda^1 \quad D_\lambda^0 = -D_p^1$$

implying

$$D^0 = f(p + \lambda, q) + g(p - \lambda, q) \quad D^1 = f(p + \lambda, q) - g(p - \lambda, q)$$

and

$$\frac{1}{2}(D_\lambda^0 D_q^1 - D_q^0 D_\lambda^1) = U(u) - V(v)$$

- Powerful constraint on f, g since λ does not enter the right side
- what is the general solution?

Note:

- Now possible to choose f, g for potentials U, V any one of sine-Gordon, Liouville, Tzitzéica, or free massive or massless.
- Tzitzéica:

$$U(u) = (e^u + 2e^{-u/2} - 3), \quad V(v) = (e^v + 2e^{-v/2} - 3)$$

and the defect potential $D^0(\lambda, p, q)$ is given by

$$D^0 = 2\sigma \left(e^{(p+\lambda)/2} + e^{-(p+\lambda)/4} \left(e^{q/2} + e^{-q/2} \right) \right) \\ + \frac{1}{\sigma} \left(8e^{-(p-\lambda)/4} + e^{(p-\lambda)/2} \left(e^{q/2} + e^{-q/2} \right)^2 \right)$$

- In sine-Gordon the type-II defect is new with two free parameters - in a sense it is two 'fused' type-I defects - [EC, Zambon \(2010\)](#)
- Other integrable field theories? - yes, for some *but not yet all*, see also [Robertson \(2014\)](#); [Bowcock, Umpleby \(2008\)](#)

Defects in quantum field theory

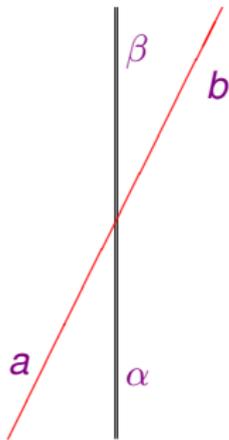
- **Expect** Soliton-defect scattering - pure transmission compatible with the bulk S-matrix
- **Expect** Topological charge will be preserved but may be exchanged with the defect
- **Expect** For each type of defect there may be several types of transmission matrix (eg in sine-Gordon expect two different transmission matrices since the topological charge on a defect can only change by ± 2 as a soliton/anti-soliton passes).
- **Expect** Not all transmission matrices need be unitary (eg in sine-Gordon one is a 'resonance' of the other)
- **Questions** Relationship between different types of defect; assemblies of defects, defect-defect scattering; fusing defects; ...

A transmission matrix is intrinsically infinite-dimensional:

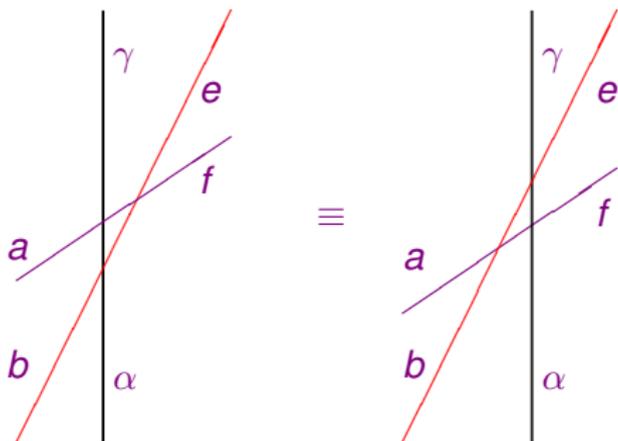
$$T_{a\alpha}^{b\beta}(\theta, \eta), \quad a + \alpha = b + \beta$$

where α, β and a, b are defect and particle (eg soliton) labels respectively (typically they will be sets of weights); and η is a collection of defect parameters.

Schematically:



Schematic compatibility relation - **Delfino, Mussardo, Simonetti (1994)**



$$S_{ab}^{cd}(\Theta) T_{d\alpha}^{f\beta}(\theta_a) T_{c\beta}^{e\gamma}(\theta_b) = T_{b\alpha}^{d\beta}(\theta_b) T_{a\beta}^{c\gamma}(\theta_a) S_{cd}^{ef}(\Theta)$$

With $\Theta = \theta_a - \theta_b$ and sums over the 'internal' indices β, c, d .

- For sine-Gordon a solution was known - **Konik, LeClair (1999)**

Zamolodchikov's sine-Gordon soliton-soliton S-matrix - reminder

$$S_{ab}^{cd}(\Theta) = \rho(\Theta) \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & C & B & 0 \\ 0 & B & C & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$$

where

$$A(\Theta) = \frac{qx_2}{x_1} - \frac{x_1}{qx_2}, \quad B(\Theta) = \frac{x_1}{x_2} - \frac{x_2}{x_1}, \quad C(\Theta) = q - \frac{1}{q}$$

$$x_a = e^{\gamma\theta_a}, \quad a = 1, 2, \quad \Theta = \theta_1 - \theta_2, \quad q = e^{i\pi\gamma}, \quad \gamma = \frac{8\pi}{\beta^2} - 1,$$

and

$$\rho(\Theta) = \frac{\Gamma(1+z)\Gamma(1-\gamma-z)}{2\pi i} \prod_1^{\infty} R_k(\Theta) R_k(i\pi - \Theta)$$
$$R_k(\Theta) = \frac{\Gamma(2k\gamma+z)\Gamma(1+2k\gamma+z)}{\Gamma((2k+1)\gamma+z)\Gamma(1+(2k+1)\gamma+z)}, \quad z = i\gamma/\pi.$$

Useful to define the variable $Q = e^{4\pi^2 i/\beta^2} = \sqrt{-q}$.

- K-L solutions have the form

$$T_{a\alpha}^{b\beta}(\theta) = f(q, x) \begin{pmatrix} Q^\alpha \delta_\alpha^\beta & q^{-1/2} e^{\gamma(\theta-\eta)} \delta_\alpha^{\beta-2} \\ q^{-1/2} e^{\gamma(\theta-\eta)} \delta_\alpha^{\beta+2} & Q^{-\alpha} \delta_\alpha^\beta \end{pmatrix}$$

where $f(q, x)$ is not uniquely determined but, for a unitary transmission matrix, should satisfy

$$\begin{aligned} \bar{f}(q, x) &= f(q, qx) \\ f(q, x)f(q, qx) &= \left(1 + e^{2\gamma(\theta-\eta)}\right)^{-1} \end{aligned}$$

- A 'minimal' solution has the following form

$$f(q, x) = \frac{e^{i\pi(1+\gamma)/4} r(x)}{1 + ie^{-2\pi iz} \bar{r}(x)},$$

where it is convenient to put $z = i\gamma(\theta - \eta)/2\pi$ and

$$r(x) = \prod_{k=0}^{\infty} \frac{\Gamma(k\gamma + 1/4 - z)\Gamma((k+1)\gamma + 3/4 - z)}{\Gamma((k+1/2)\gamma + 1/4 - z)\Gamma((k+1/2)\gamma + 3/4 - z)}$$

$$T_{a\alpha}^{b\beta}(\theta) = f(q, x) \begin{pmatrix} Q^\alpha \delta_\alpha^\beta & q^{-1/2} e^{\gamma(\theta-\eta)} \delta_\alpha^{\beta-2} \\ q^{-1/2} e^{\gamma(\theta-\eta)} \delta_\alpha^{\beta+2} & Q^{-\alpha} \delta_\alpha^\beta \end{pmatrix}$$

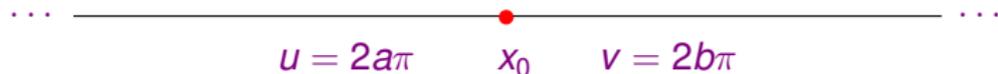
Remarks (supposing $\theta > 0$) - **Bowcock, EC, Zambon (2005)**:

Tempting to suppose η (possibly renormalized) is the same parameter as in the type I classical model.

- $\eta < 0$ - the off-diagonal entries dominate;
 - $\theta > \eta > 0$ - the off-diagonal entries dominate;
 - $\eta > \theta > 0$ - the diagonal entries dominate.
- Similar features to the classical soliton-defect scattering.
 - The different behaviour of solitons versus anti-solitons (diagonal terms) is a direct consequence of the defect term proportional to

$$\delta(x - x_0)(uv_t - vu_t)/2$$

Consider the x-axis with a defect located at x_0 and asymptotic values of the fields



A horizontal line representing the x-axis. A red dot is placed on the line, labeled x_0 below it. To the left of the dot, the text $u = 2a\pi$ is written. To the right of the dot, the text $v = 2b\pi$ is written. Ellipses (\dots) are placed at both ends of the line.

Compare $(0, 0)$ and (a, b) in functional integral representation:

$$u \rightarrow u - 2a\pi, \quad v \rightarrow v - 2b\pi, \quad A \rightarrow A + \delta A$$

with

$$\delta A = \frac{\pi}{\beta^2} \int_{-\infty}^{\infty} dt (av_t - bu_t) = \frac{\pi}{\beta^2} (a\delta v - b\delta u)_{x_0}$$

So

$$T(a, b) = e^{i\pi(a\delta v - b\delta u)/\beta^2} T(0, 0)$$

Also

- soliton to soliton: $(a, b) \rightarrow (a - 1, b - 1)$, so $\delta u = \delta v = -2\pi$

- anti-soliton to anti-soliton: $(a, b) \rightarrow (a + 1, b + 1)$, so $\delta u = \delta v = 2\pi$

which leads to relative changes of phase

$$e^{\pm 2i\pi^2(b-a)/\beta^2} = Q^{\pm\alpha/2}, \quad \alpha = b - a$$

Demonstrates the difference in the diagonal elements, though there is also a phase factor in the off-diagonal terms.

Removing this by a unitary change of basis leads to the diagonal terms provided above.

- $\theta = \eta$ is not special (neither is $z = -1/4$) but there is a simple pole nearby at $z = 1/4$:

$$\theta = \eta - \frac{i\pi}{2\gamma} \rightarrow \eta, \text{ as } \beta \rightarrow 0$$

This pole is like a resonance, with complex energy,

$$E = m_s \cosh \theta = m_s(\cosh \eta \cos(\pi/2\gamma) - i \sinh \eta \sin(\pi/2\gamma))$$

and a 'width' proportional to $\sin(\pi/2\gamma)$.

- The Zamolodchikov S-matrix has 'breather' poles corresponding to soliton-anti-soliton bound states at

$$\Theta = i\pi(1 - n/\gamma), \quad n = 1, 2, \dots, n_{\max};$$

use the bootstrap to define the transmission factors for breathers and find for the lightest breather:

$$T(\theta) = -i \frac{\sinh\left(\frac{\theta-\eta}{2} - \frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta-\eta}{2} + \frac{i\pi}{4}\right)}$$

Type II transmission matrix for sine-Gordon - EC, Zambon (2010)

There is another, more general, set of solutions to the quadratic relations for the transmission matrix:

$$\rho(\theta) \begin{pmatrix} (a_+ Q^\alpha + a_- Q^{-\alpha} x^2) \delta_\alpha^\beta & x (b_+ Q^\alpha + b_- Q^{-\alpha}) \delta_\alpha^{\beta-2} \\ x (c_+ Q^\alpha + c_- Q^{-\alpha}) \delta_\alpha^{\beta+2} & (d_+ Q^\alpha x^2 + d_- Q^{-\alpha}) \delta_\alpha^\beta \end{pmatrix}$$

where $x = e^{\gamma\theta}$.

The free constants satisfy the two constraints

$$a_\pm d_\pm - b_\pm c_\pm = 0$$

These and $\rho(\theta)$ are constrained further by crossing and unitarity.

- For a choice of parameters this describes a type II defect.
- With $a_- = d_+ = 0$ and $b_+ = c_- = 0$ or $b_- = c_+ = 0$ (after a similarity transformation), reduces to the type I solution.
- For certain other choices of parameters it reduces to a direct sum of the Zamolodchikov S-matrix and two infinite dimensional pieces.

Alternative formulation - Weston (2010)

Summary: for Type II

$$T = \rho(x) \begin{pmatrix} xa_+ Q^{-N} + x^{-1} a_- Q^N & a \\ a^* & xd_+ Q^N + x^{-1} d_- Q^{-N} \end{pmatrix},$$

where a^* and a are 'generalised' raising and lowering operators, respectively,

$$a^*|k\rangle = |k+2\rangle \quad a|k\rangle = F(k)|k-2\rangle \quad N|k\rangle = k|k\rangle, \quad k \in \mathbb{Z}$$

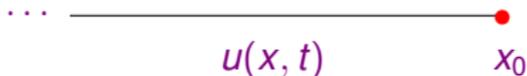
$$F(N) = f_0 + f_+ Q^{2N} + f_- Q^{-2n}, \quad f_+ = Q^{-2} a_- d_+ \quad f_- = Q^2 a_+ d_-$$

- T intertwines the coproducts of a finite (soliton) and infinite (defect) representations of the Borel subalgebra of $U_q(\mathfrak{a}_1^{(1)})$.

- Idea generalises to all other quantum algebras allowing (in principle) calculations of associated defect matrices. For some examples see [EC, Zambon \(2010\)](#), [Boos et al. \(2011\)](#).

Boundaries revisited: the sine-Gordon model EC, Zambon (2012)

Start with a single selected point on the x -axis, say $x_0 = 0$, and denote the field to the left ($x < 0$) by u :



- The sine-Gordon model with a general (two-parameter) integrable boundary condition was analyzed by Ghoshal, Zamolodchikov (1994)...
- ...and sine-Gordon model with dynamical boundary - Baseilhac, Delius (2001), Baseilhac, Koizumi (2003)
- A defect (or several defects) can be placed in front of the boundary and generate a new boundary (as seen from $x \ll 0$); for the sinh-Gordon example, see Bajnok, Simon (2008).

But...

- The defect will introduce dependence on topological charge in the modified reflection matrix.
- Generally, the boundary should be considered as carrying topological charge, which may change as a soliton reflects.
- Ansatz

$$R_{a\alpha}^{b\beta}(\theta) = \begin{pmatrix} r_+(\alpha, \mathbf{x}) \delta_\alpha^\beta & s_+(\alpha, \mathbf{x}) \delta_\alpha^{\beta-2} \\ s_-(\alpha, \mathbf{x}) \delta_\alpha^{\beta+2} & r_-(\alpha, \mathbf{x}) \delta_\alpha^\beta \end{pmatrix}$$

- Boundary Yang-Baxter equation **Cherednik (1984)**

$$R_{a\alpha}^{q\beta}(\theta_a) S_{bq}^{ps}(\Theta_+) R_{p\beta}^{r\gamma}(\theta_b) S_{sr}^{dc}(\Theta_-) = S_{ba}^{pq}(\Theta_-) R_{p,\alpha}^{r\beta}(\theta_b) S_{qr}^{sc}(\Theta_+) R_{s\beta}^{d\gamma}(\theta_a),$$

with $\Theta_+ = (\theta_b + \theta_a)$ and $\Theta_- = (\theta_b - \theta_a)$.

- Ghoshal-Zamolodchikov solution reformulated

$$R_{a\alpha}^{b\beta}(\theta) = \sigma(\theta) \begin{pmatrix} (r_1 x + r_2/x) \delta_\alpha^\beta & k_0 (x^2 - 1/x^2) \delta_\alpha^{\beta-2} \\ l_0 (x^2 - 1/x^2) \delta_\alpha^{\beta+2} & (r_2 x + r_1/x) \delta_\alpha^\beta \end{pmatrix}$$

and $l_0 = k_0$, $r_1 r_2 = 1$.

- General solution (we think)

$$r_+(\alpha, x) = (x^2 - 1/x^2) (r_3 q^{\alpha+1} x - r_4 q^{-\alpha-1}/x) + r_1 x + r_2/x,$$

$$r_-(\alpha, x) = (x^2 - 1/x^2) (r_4 q^{-\alpha+1} x - r_3 q^{\alpha-1}/x) + r_2 x + r_1/x,$$

$$s_+(\alpha, x) = (x^2 - 1/x^2) (k_0 + k_1 q^\alpha + k_2 q^{-\alpha}),$$

$$s_-(\alpha, x) = (x^2 - 1/x^2) (l_0 + l_1 q^\alpha + l_2 q^{-\alpha}),$$

$$k_1 l_1 = -r_3^2, \quad k_2 l_2 = -r_4^2, \quad k_1 l_0 + q^2 k_0 l_1 = q r_2 r_3, \quad k_0 l_2 + q^2 k_2 l_0 = q r_1 r_4.$$

- A defect placed in front of a boundary generalises R according to

$$R_{a\alpha}^{b\beta} \tilde{\beta}_{\tilde{\alpha}}(\theta) = T_{a\tilde{\alpha}}^{c\tilde{\gamma}}(\theta) R_{c\alpha}^{d\beta}(\theta) \hat{T}_{d\tilde{\gamma}}^{b\tilde{\beta}}(\theta)$$

where $\hat{T}(\theta) = T^{-1}(-\theta)$.

Begin with an R matrix corresponding to a Dirichlet boundary condition,

$$R^{(0)}_{c\alpha}{}^{d\beta}(\theta) = \sigma(\theta) \begin{pmatrix} (rx + x^{-1}r^{-1})\delta_{\alpha}^{\beta} & 0 \\ 0 & (rx^{-1} + xr^{-1})\delta_{\alpha}^{\beta} \end{pmatrix}$$

- $T_{II}R^{(0)}\hat{T}_{II}$ is equivalent to the general solution given above when T_{II} is the general type II transmission matrix;
- $T_I R^{(0)}\hat{T}_I$ equivalent to the G-Z solution when T_I is restricted to the type I (Konik-LeClair) transmission matrix.
- Is there a Lagrangian description of the generalised boundary condition corresponding to the general solution? For example

$$\mathcal{L}_B(u, \lambda) = \theta(-x) \mathcal{L}_{sG} + \delta(x)(u\lambda_t - B(u, \lambda)),$$

with

$$B(u, \lambda) = e^{\lambda/2}f(u) + e^{-\lambda/2}g(u),$$

and

$$f(u)g(u) = h_+ e^{u/2} + h_- e^{-u/2} + 2(e^u + e^{-u}) + h_0$$

Defect-defect scattering - type I

$$T_{1 a\alpha}^{b\gamma} T_{2 b\beta}^{c\delta} U_{\gamma\delta}^{\rho\sigma} = U_{\alpha\beta}^{\delta\gamma} T_{2 a\delta}^{b\rho} T_{1 b\gamma}^{c\sigma}.$$

$$T_i = \begin{pmatrix} \alpha_i Q^{N_i} & \sqrt{q} \beta_i x A_i \\ \sqrt{q} \beta_i x A_i^* & \alpha_i^{-1} Q^{-N_i} \end{pmatrix}, \quad i = 1, 2$$

where

$$x = e^{\gamma\theta}, \quad q = e^{i\pi\gamma}, \quad Q^2 = -q; \quad |\alpha| = 1, \quad \beta^* = \beta.$$

Data carried by $\alpha_i, \beta_i, A_i, A_i^*, i = 1, 2$ with two sets of mutually commuting annihilation and creation operators.

U is independent of x : equating terms in powers of x leads to the following four equations:

$$Q^{N_1+N_2} U = U Q^{N_1+N_2}, \quad A_1 U A_1 = A_2 U A_2$$

$$\left(\alpha_1 \beta_2 Q^{N_1} A_2 + \alpha_2^{-1} \beta_1 Q^{-N_2} A_1 \right) U = U \left(\alpha_2 \beta_1 Q^{N_2} A_1 + \alpha_1^{-1} \beta_2 Q^{-N_1} A_2 \right)$$

$$\left(\alpha_2 \beta_1 Q^{N_2} A_1^* + \alpha_1^{-1} \beta_2 Q^{-N_1} A_2^* \right) U = U \left(\alpha_1 \beta_2 Q^{N_1} A_2^* + \alpha_2^{-1} \beta_1 Q^{-N_2} A_1^* \right)$$

$$U = \sum_{k=-\infty}^{\infty} A_1^k A_2^{-k} U_k(N_1, N_2, \lambda), \quad \lambda = \beta_1/\beta_2$$

Then

$$U_{k+2}(N_1, N_2, \lambda) = U_k(N_1 - 2, N_2 + 2, \lambda)$$

$$U_{2l}(N_1, N_2, \lambda) = U_0(N_1 - 2l, N_2 + 2l, \lambda)$$

$$U_{2l+1}(N_1, N_2, \lambda) = U_1(N_1 - 2l, N_2 + 2l, \lambda).$$

and

$$(\lambda Q^{-N_2} A_1 + Q^{N_1} A_2) U = U (Q^{-N_1} A_2 + \lambda Q^{N_2} A_1),$$

$$Q^{N_1-2} U_1^{(N_1, N_2)} + \lambda Q^{-N_2} U_0^{(N_1, N_2)} = \lambda Q^{N_2} U_0^{(N_1-2, N_2)} + Q^{-N_1} U_1^{(N_1, N_2-2)}$$

$$Q^{N_1} U_0^{(N_1, N_2+2)} + \lambda Q^{-N_2} U_1^{(N_1+2, N_2)} = \lambda Q^{N_2+2} U_1^{(N_1, N_2)} + Q^{-N_1} U_0^{(N_1, N_2)}$$

Formal generating functionals

$$U(x, y) = \sum_{n,m} x^n y^m U_0(n, m), \quad V(x, y) = \sum_{n,m} x^n y^m U_1(n, m)$$

Then

$$\begin{aligned} \lambda U(x, y/Q) + Q^{-2} V(Qx, y) &= \lambda x^2 U(x, Qy) + y^2 V(x/Q, y) \\ (\lambda/x^2) V(x, y/Q) + (1/y^2) U(Qx, y) &= \lambda Q^2 V(x, Qy) + U(x/Q, y). \end{aligned}$$

These can be written slightly more symmetrically by rearranging and putting $r = Q$, $s = \lambda Q$:

$$\begin{aligned} x U(x, ry) - x^{-1} U(x, r^{-1}y) &= \frac{y}{xs} ((ry)^{-1} V(rx, y) - ry V(r^{-1}x, y)) \\ y^{-1} U(rx, y) - y U(r^{-1}x, y) &= \frac{ys}{x} (rx V(x, ry) - r(x)^{-1} V(x, r^{-1}y)). \end{aligned}$$

- What is the general solution?

Further questions....

- Some alternative views and other aspects are discussed in several places. Eg - [Habibullin, Kundu \(2008\)](#); [Bajnok, Simon \(2008\)](#)
- Other Toda models - defects can be constructed for some other affine Toda models, eg the $a_r^{(1)}$, $(c_n^{(1)}, d_{n+1}^{(2)})$, $a_{2n}^{(2)}$

[Bowcock, EC, Zambon \(2004\)](#), [EC, Zambon \(2007, 2010, 2011\)](#), [Robertson \(2014\)](#).

What about all the others?

- Once the question is answered....
-What are the general integrable boundary conditions for all affine Toda field theories?