Integrability from Defects

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A few coincidences....

- First degree 1968
- PhD 1972
- Developed interest in integrable systems/skyrmions, ∼1990
- Hollowood and Miramontes

and a thank you...

- ‘Integrability, Non-Perturbative Effects and Symmetry in Quantum Field Theory’,15-20 September 1997
- ‘Integrable Models and Applications’, 12-16 September 2005

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Introducing a 'defect' or 'discontinuity' - Bowcock, EC, Zambon (2003)

Start with a single selected point on the x-axis, say x_0, and denote the field to the left ($x < x_0$) by u, and to the right ($x > x_0$) by v:

$$\cdots \quad u(x,t) \qquad x_0 \qquad v(x,t) \quad \cdots$$

Relativistic scalar field equations in separated domains:

$$\partial^2 u = -\frac{\partial U}{\partial u}, \quad x < x_0, \quad \partial^2 v = -\frac{\partial V}{\partial v}, \quad x > x_0$$

- How can the fields u, v be 'sewn' together at x_0?
- Where might integrability come from?
- One natural choice (δ-impurity) would be to put

$$u(x_0,t) = v(x_0,t), \quad u_x(x_0,t) - v_x(x_0,t) = F(u(x_0,t)),$$

- studied numerically - Goodman, Holmes, Weinstein 2002

• Problem: there is a distinguished point, translation symmetry is lost and conservation laws - at least some of them - (for example, momentum), are violated unless the defect contributes compensating terms via sewing conditions.

Consider the field contributions to energy-momentum:

$$P^\mu = \int_{-\infty}^{x_0} dx\, T^{0\mu}(u) + \int_{x_0}^{\infty} dx\, T^{0\mu}(v).$$

Using the field equations, can we arrange

$$\frac{dP^\mu}{dt} = -\left[T^{1\mu}(u)\right]_{x_0} + \left[T^{1\mu}(v)\right]_{x_0} = -\frac{dD^\mu(u,v)}{dt}$$

with the right hand side depending only on the fields at x_0?

If so, $P^\mu + D^\mu$ is conserved with D^μ being the defect contribution.

• Only a few possible sewing conditions (and bulk potentials $U,\ V$) are permitted for this to work. To see why...

Consider the field contribution to energy and calculate

$$\frac{dP^0}{dt} = [u_x u_t]_{x_0} - [v_x v_t]_{x_0}.$$

Setting

$$u_x = v_t + X(u,v), \quad v_x = u_t + Y(u,v)$$

we find

$$\frac{dP^0}{dt} = u_t X - v_t Y.$$

This is a total time derivative if

$$X = -\frac{\partial D^0}{\partial u}, \quad Y = \frac{\partial D^0}{\partial v},$$

for some D^0. Then

$$\frac{dP^0}{dt} = -\frac{dD^0}{dt}.$$

- Expected anyway since time translation remains good.

On the other hand, for momentum

$$\frac{dP^1}{dt} = -\left[\frac{u_t^2 + u_x^2}{2} - U(u)\right]_{x_0} + \left[\frac{v_t^2 + v_x^2}{2} - V(v)\right]_{x_0}$$

$$= \left[-v_t X + u_t Y - \frac{X^2 - Y^2}{2} + U - V\right]_{x_0} = -\frac{dD^1(u,v)}{dt}$$

This is a total time derivative provided the first piece is a perfect differential and the second piece vanishes. Thus

$$X = -\frac{\partial D^0}{\partial u} = \frac{\partial D^1}{\partial v}, \quad Y = \frac{\partial D^0}{\partial v} = -\frac{\partial D^1}{\partial u},$$

In other words

$$\frac{\partial^2 D^0}{\partial v^2} = \frac{\partial^2 D^0}{\partial u^2}, \quad \frac{1}{2}\left(\frac{\partial D^0}{\partial u}\right)^2 - \frac{1}{2}\left(\frac{\partial D^0}{\partial v}\right)^2 = U(u) - V(v).$$

Highly constraining - just a few possible combinations for U, V, D^0

- sine-Gordon, Liouville, massless free, or, massive free.

For example, if $U(u) = m^2u^2/2,\ V(v) = m^2v^2/2$, D^0 turns out to be

$$D^0(u, v) = \frac{m\sigma}{4}(u + v)^2 + \frac{m}{4\sigma}(u - v)^2,$$

and σ is a free parameter.

- Note: the Tzitzéica (aka BD, MZS, $a_2^{(2)}$ affine Toda) potential

$$U(u) = e^u + 2e^{-u/2}$$

is **not** possible.

- There is a Lagrangian description of this type of defect (type I):

$$\mathcal{L} = \theta(x_0 - x)\mathcal{L}(u) + \delta(x - x_0)\left(\frac{uv_t - u_tv}{2} - D^0(u, v)\right) + \theta(x - x_0)\mathcal{L}(v)$$

sine-Gordon - Bowcock, EC, Zambon (2003, 2004, 2005)

Choosing u, v to be sine-Gordon fields (and scaling the coupling and mass parameters to unity), the allowed possibilities are:

$$D^0(u,v) = -2\left(\sigma \cos\frac{u+v}{2} + \sigma^{-1}\cos\frac{u-v}{2}\right),$$

where σ is a constant, to find

$$\begin{aligned}
x < x_0: \quad \partial^2 u &= -\sin u, \\
x > x_0: \quad \partial^2 v &= -\sin v, \\
x = x_0: \quad u_x &= v_t - \sigma \sin\frac{u+v}{2} - \sigma^{-1}\sin\frac{u-v}{2}, \\
x = x_0: \quad v_x &= u_t + \sigma \sin\frac{u+v}{2} - \sigma^{-1}\sin\frac{u-v}{2}.
\end{aligned}$$

- The final two are a Bäcklund transformation frozen at $x = 0$.
- The defect could be anywhere - essentially topological
- Higher conserved charges and other properties also fine.

Classical type II defect - EC, Zambon (2009)

Consider two relativistic field theories with fields u and v, and add a new degree of freedom $\lambda(t)$ at the defect location ($x_0 = 0$):

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left((u-v)\lambda_t - D^0(\lambda, u, v)\right)$$

Then the usual Euler-Lagrange equations lead to

- equations of motion:

$$\partial^2 u = -\frac{\partial U}{\partial u} \quad x < 0, \qquad \partial^2 v = -\frac{\partial V}{\partial v} \quad x > 0$$

- defect conditions at $x = 0$:

$$u_x = \lambda_t - D^0_u \qquad v_x = \lambda_t + D^0_v \qquad (u-v)_t = -D^0_\lambda.$$

As before, consider momentum

$$P^1 = -\int_{-\infty}^{0} dx\, u_t u_x - \int_0^{\infty} dx\, v_t v_x,$$

and seek a functional $D^1(u, v, \lambda)$ such that $P^1_t \equiv -D^1_t$.

Implies constraints on $U,\ V,\ D^1$.

Putting $q = (u - v)/2,\ p = (u + v)/2$ these are:

$$D^0_p = -D^1_\lambda \qquad D^0_\lambda = -D^1_p$$

implying

$$D^0 = f(p + \lambda, q) + g(p - \lambda, q) \qquad D^1 = f(p + \lambda, q) - g(p - \lambda, q)$$

and

$$\frac{1}{2}(D^0_\lambda D^1_q - D^0_q D^1_\lambda) = U(u) - V(v)$$

- Powerful constraint on $f,\ g$ since λ does not enter the right side - what is the general solution?

Note:

- Now possible to choose f, g for potentials U, V any one of sine-Gordon, Liouville, Tzitzéica, or free massive or massless.
- Tzitzéica:

$$U(u) = (e^{u} + 2\,e^{-u/2} - 3), \quad V(v) = (e^{v} + 2\,e^{-v/2} - 3)$$

and the defect potential $D^0(\lambda, p, q)$ is given by

$$\begin{aligned} D^0 = \quad & 2\sigma \left(e^{(p+\lambda)/2} + e^{-(p+\lambda)/4} \left(e^{q/2} + e^{-q/2} \right) \right) \\ & + \frac{1}{\sigma} \left(8\, e^{-(p-\lambda)/4} + e^{(p-\lambda)/2} \left(e^{q/2} + e^{-q/2} \right)^2 \right) \end{aligned}$$

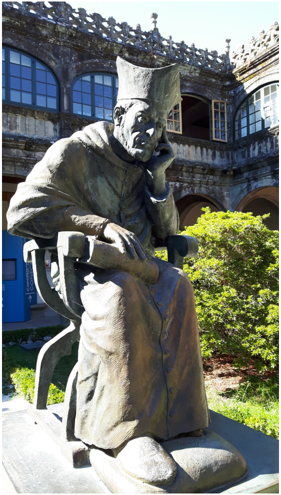
- In sine-Gordon the type-II defect is new with two free parameters - in a sense it is two 'fused' type-I defects - EC, Zambon (2010)

Generalisations

- Multi-component fields - what about other affine Toda field theories?

  - $a_n^{(1)}$ affine Toda with type I - EC, Zambon (2009)

  - type II defects can be constructed for some other affine Toda models, eg the $a_r^{(1)},\ (c_n^{(1)},\ d_{n+1}^{(2)}),\ a_{2n}^{(2)}$ - Bowcock, EC, Zambon (2004), EC, Zambon (2007, 2010, 2011), Robertson (2014).
- What about nonlinear Schrödinger, KdV, mKdV, etc, etc? newline - yes, see Caudrelier, Mintchev, Ragoucy (2004,) EC, Zambon (2005), Caudrelier (2008), . . .
- Is the setup genuinely integrable? For an alternative (algebraic) approach see Avan, Doikou (2012, 2013); Doikou (2014, 2016)
- What about SUSY? See, for example, Gomes, Ymai, Zimerman (2008); Aguirre, Gomes, Spano, Zimerman (2015)



Buena suerte, Joaquín