Higher limits of sheaves

Brent Everitt – York
(joint with Paul Turner – Geneva)

- P = (fixed) small category
- F sheaf on P = (covariant) functor $P^{op} \to {}_R\mathrm{Mod}$

commutative with 1

- Can form the modules:

$$\varprojlim_P F \qquad\qquad \varinjlim^P F$$

General nonsense

- P = (fixed) small category
- F sheaf on P = (covariant) functor $P^{op} \to {}_R\mathrm{Mod}$

commutative with 1

- Can form the modules, but the functors:

$$\varprojlim_P F \leftsquigarrow F \rightsquigarrow \varinjlim^P F$$

are not exact

- Hence:

right and left derived functors

$$H^i(P;F) =: \varprojlim_P^i F \qquad \varinjlim_i^P F := H_i(P;F)$$

Cohomology of P with coefficients in F

Homology of P with coefficients in F

Constant sheaves (= just the topology of P)

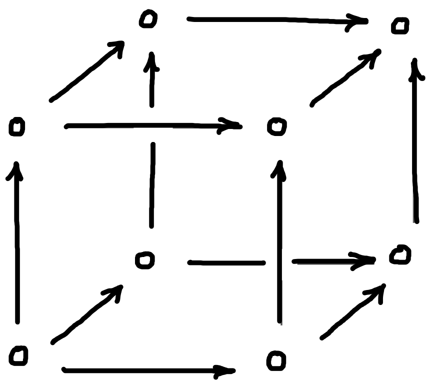
- $A \in {}_R\mathrm{Mod}$ $\quad$ ΔA = constant sheaf

  $\Rightarrow H^*(P;\Delta A) \cong H^*(BP;A)$ ordinary singular homolgy

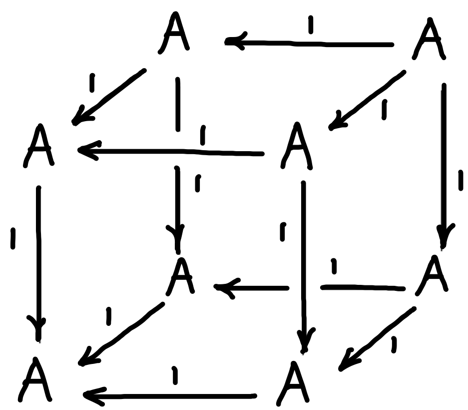
  classifying space

- E.g.: P = poset

minimum $\underline{0}$

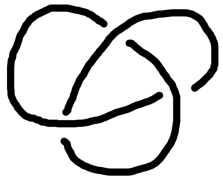


$\rightsquigarrow \Delta A =$

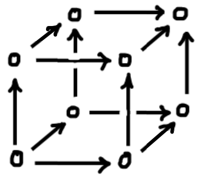


BP = cone on $B(P \setminus \underline{0}) \simeq * \Rightarrow H^i(P;\Delta A) = 0$, $i>0$

- E.g.:

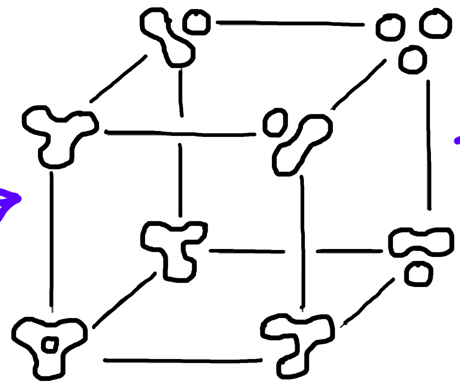


D = link diagram



P = subsets of the crossings

cube of smoothings

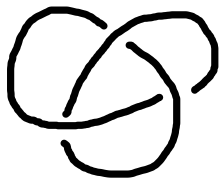


F = Khovanov sheaf

$$\begin{tikzcd}
 & A^{\otimes 2}\{1\} \arrow[dl, "m"'] \arrow[dd, "m"] & & A^{\otimes 3} \arrow[ll, "m"'] \arrow[dl, "m"] \arrow[dd, "m"] \\
A\{2\} \arrow[dd, "\Delta"'] & & A^{\otimes 2}\{1\} \arrow[ll, "m"'] \arrow[dd, "m"] & \\
 & A\{2\} \arrow[dl, "\Delta"] & & A^{\otimes 2}\{1\} \arrow[ll, "m"'] \arrow[dl, "m"] \\
A^{\otimes 2}\{3\} & & A\{2\} \arrow[ll, "\Delta"] &
\end{tikzcd}$$

(A = a certain graded module)

- E.g.:



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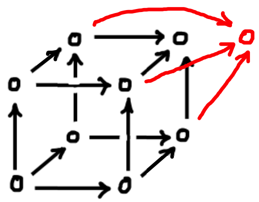

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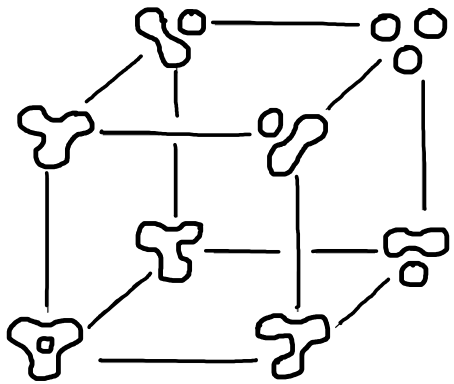

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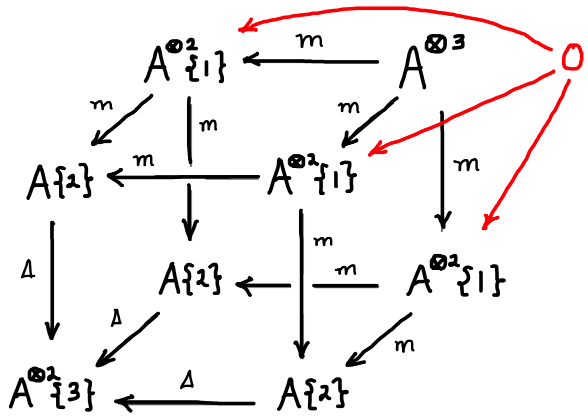
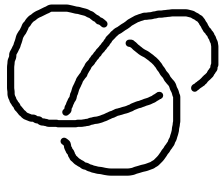


(A = a certain graded module)

- E.g.:



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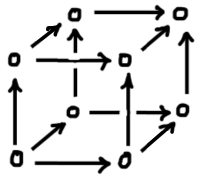

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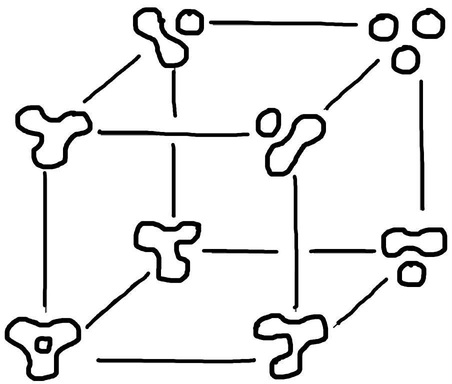

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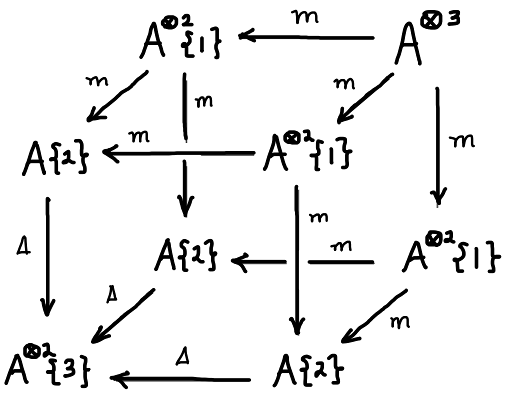


(A = a certain graded module)

- Theorem (E.-Turner '14)

$$H^{*,*}(P;F) \cong \overline{Kh}^{*,*}(D)$$ (unnormalised Khovanov homology)

($\Rightarrow \chi H(P;F) = \langle D \rangle$ Kauffman bracket)

- moral: P trivial topologically; but $H(P;F)$ highly non-trivial

The sheaf does all the work # 2

reductive algebraic group $/\mathbb{C}$

- G/B flag variety

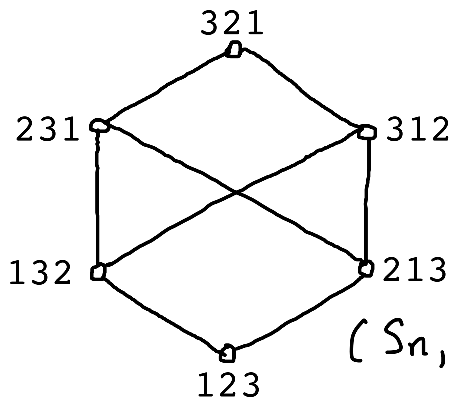
Borel subgroup

$GL_n\mathbb{C}/\left\{\begin{pmatrix} & * \\ 0 & \end{pmatrix}\right\}$

flag manifold

- P = Schubert varieties ordered by $\subseteq$

  ($\cong$ Weyl group with Bruhat order)



$(S_n, \text{Bruhat} \leq)$

BP trivial topologically

Schubert variety

- <u>sheaf</u> $F(x) = \bigoplus_{x \leq y} \mathcal{I}\mathcal{H}^*(\overline{C(y)})_x$ (Intersection cohomology)

$\Rightarrow \chi H^*(P \setminus \underline{1}\,; F) = (-1)^{\ell(w_0)}(KL - 1) + 1$

Kazhdan-Lusztig polynomial of W

- V finite dim. / k

  $\mathcal{A}$ = finite # linear hyperplanes $\subset V$ ← arrangement

  P = all $\bigcap$'s under $\supseteq$ ← intersection lattice

- P has a max $\underline{1}$ and min $\underline{0}$

  $$\Rightarrow BP \simeq *$$

  $$B(P \setminus \underline{1}, \underline{0}) \simeq \bigvee^m S^n$$

  $$n = \dim \underline{0} - \dim \underline{1} - 2$$
  $$m = \mu(\underline{0}, \underline{1})$$

  [Folkman-Quillen]

- natural sheaf F on P:

  $F(x) = x \quad F(x \supseteq y) = F(x) \hookleftarrow F(y)$

- Theorem [Lusztig '74]

  k finite, $\mathcal{A}$ = all hyperplanes

  $$\Rightarrow H_i(P \setminus \underline{0}, \underline{1}; F) \cong \begin{cases} V, & i=0 \\ 0, & 0<i<\dim \underline{1}-2 \end{cases}$$

- Theorem [E.-Turner '17]

  any k, $\mathcal{A}$ = "braid" arrangement ← reflecting hyperplanes of S_n

  $$\Rightarrow H^i(P \setminus \underline{1}; F) \cong \begin{cases} \bigcap \mathcal{A}, & i=0 \\ V/H, & i=\dim \underline{1}-1 \\ 0, & \text{else} \end{cases}$$

  a hyperplane → H