The sympathetic sceptics guide to semigroup representations

Brent Everitt⋆

Abstract. version August 30, 2016

BRENT EVERITT: Department of Mathematics, University of York, York YO10 5DD, United Kingdom. e-mail: brent.everitt@york.ac.uk.

⋆ Based on lectures given at York in the Spring of 2016. I am grateful to Michael Bate, John Fountain and Vicky Gould for a number of helpful suggestions and improvements.

Semigroup representations (Spring 2016)

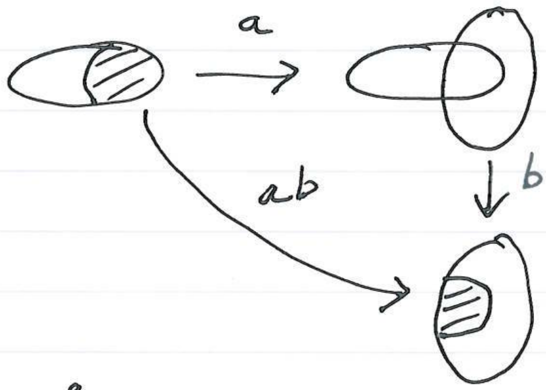
1. Semigroup basics

- S = finite semigp.
- (three running) Eg's: $[n]=\{1,\dots,n\}$

(i). S_n = all bijections $[n]\xrightarrow{a}[n]$ under composition.

(ii). I_n = all partial bijections $X\xrightarrow{a}Y$, $X,Y\subseteq[n]$

under composition:



(all fcns, actions, etc on the right)

(ii). T_n = all maps $[n]\xrightarrow{a}[n]$ under composition.

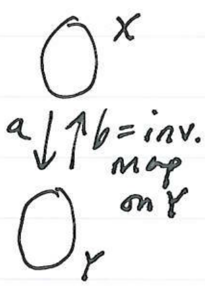
- inverses in semi-gps: an inverse of $a\in S$ is a b s.t

$$(*)\quad \begin{array}{l} aba=a \\ bab=b \end{array}$$

(i). S_n a monoid s.t $\forall a\ \exists!\, b$ with $ab=1=ba$ ($\Rightarrow(*)$)

i.e: a gp!

(ii). I_n a monoid s.t. $\forall a\ \exists!\, b$ satisfying $(*)$



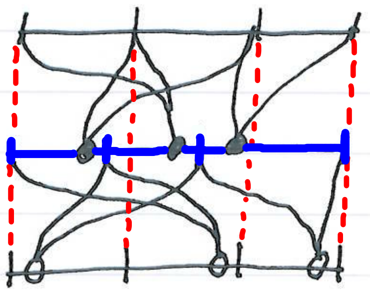
($\Rightarrow$ $ab=id_X$ idempotents)
$ba=id_Y$

i.e: I_n an inverse monoid

(iii). T_n a monoid s.t $\forall a\ \exists$ (many) b satisfying $(*)$

fibres of a (= equiv. classes of ker a)

e.g.:



T_n a regular monoid

from now on S = finite regular monoid

• Structure : Green's relations in S_n, I_n, T_n

1. $a\mathcal{L}b \iff im(a) = im(b)$

2. $a\mathcal{R}b \iff$ fibres of a = fibres of b

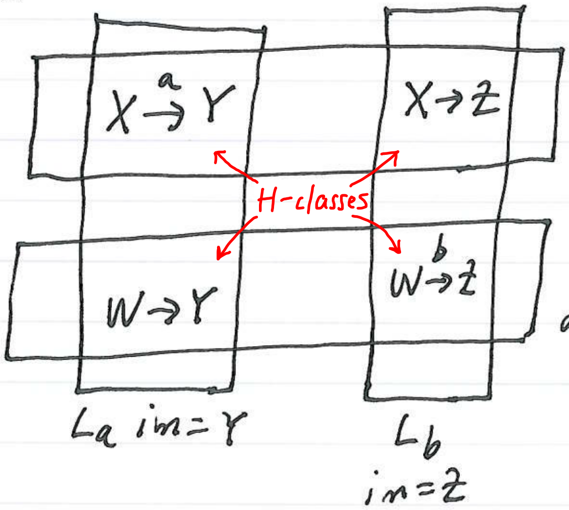
$$\left(\overset{S_n}{\underset{I_n}{\iff}} dom(a) = dom(b) \right)$$

3. $a\mathcal{J}b \iff |im(a)| = |im(b)|$

4. $a\mathcal{H}b \iff 1 + 2.$

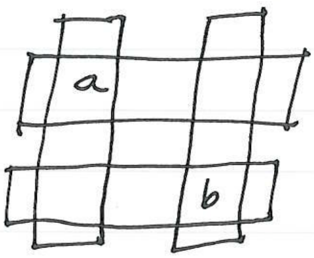
S_n: these are all trivial! (any a, b are related)

I_n: (by 1, 2)



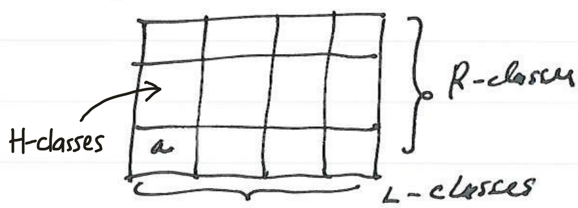
R and L commute to give nice "eggbox" picture

by 3: $a\mathcal{J}b \iff$



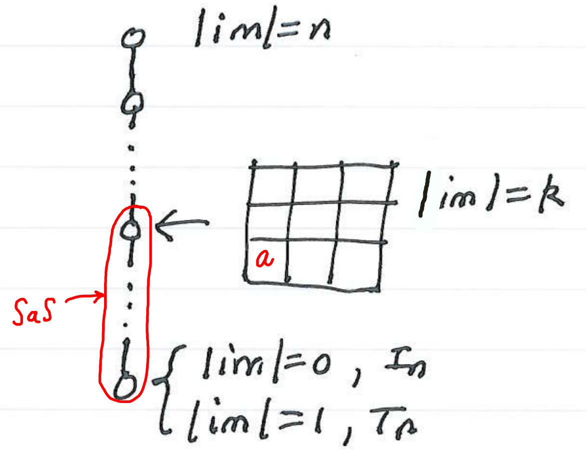
belong in same eggbox

i.e.: J_a = J-class of a:



These pictures hold for all S (finite regular monoids).

In I_n/T_n the J-classes naturally ordered:



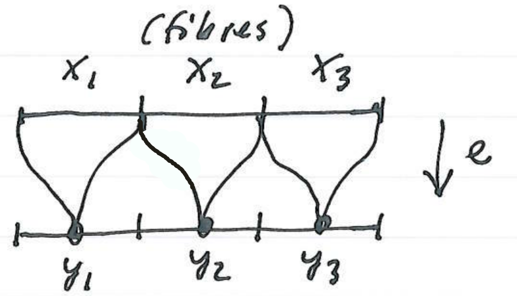
In general, $J_a \leq J_b$ $\overset{\text{def}}{\iff}$ $SaS \subseteq SbS$ partial order.

- <u>Idempotents / subgroups</u>: idempotent $e = e^2$

I_n: $e : X \xrightarrow{id_X} X$, $X \subseteq [n]$

Fix domain X: only one such map $\Rightarrow$ every R-class has exactly one idempotent (similarly every L-class)

T_n:

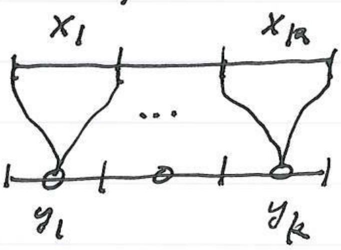


$e : X_i \mapsto y_i$ with $y_i \in X_i$

Fix the fibres and wiggle the y_i inside them (or conversely) $\Rightarrow$ every R-class has at least one idem. (similarly L-classes).

Similarly for every inverse or regular semigp

H-class of e in T_n



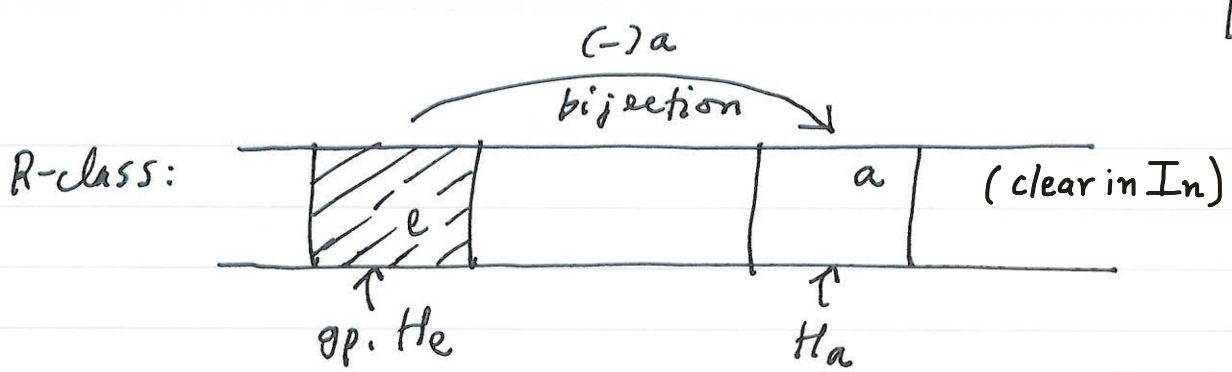
contains all bijections $\{x_1, \dots, x_k\} \to \{y_1, \dots, y_k\}$

i.e: a copy of S_k in T_n

In general H_e a <u>subgp.</u> of S (with identity e)

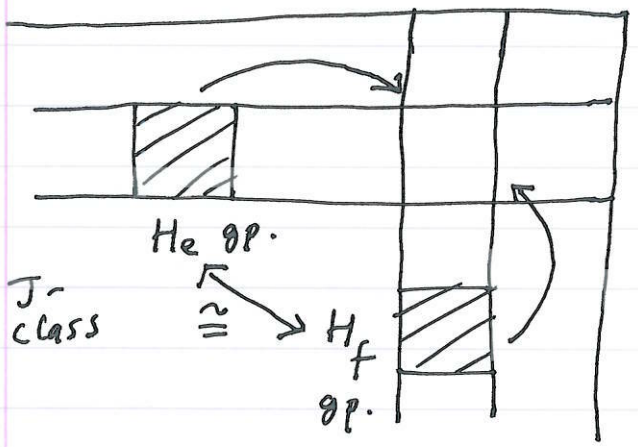
If G a subgroup of S then $G \subseteq H_e$ for some e.

(hence the H_e are <u>maximal</u> subgroups)



$\Rightarrow$ every element of H_a has unique expression ga ($g \in H_e$)

(and similarly in an L-class).



Two gp. H-classes in a J-class isomorphic.

2. Representations basics (S = finite regular monoid)

- k = field, V = finite dim. vector space $/k$.

$\mathrm{End}(V)$ = monoid of all linear maps $V \to V$ under composition

An <u>S-action</u> or <u>linear representation of S</u> a (monoid) homom.

$$S \xrightarrow{\varphi} \mathrm{End}(V). \qquad (\Rightarrow 1_S \overset{\varphi}{\mapsto} \mathrm{id} \in \mathrm{End}(V))$$

<u>note</u>: (i) $\mathrm{im}\,\varphi \neq \{0\}$. (ii). If S a group then $\mathrm{im}\,\varphi \subseteq GL(V)$ = gp. invertible linear maps $V \to V$.

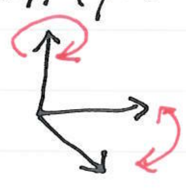
<u>abuses</u>: identify $a \in S$ and $(a)\varphi \in \mathrm{End}(V)$; for $v \in V$ write $v \cdot a$ or va for effect of $(a)\varphi$ on v; say V is an S-representation.

- <u>Eg</u> (mapping representations): V = k-space with basis $\{v_1, \ldots, v_n\}$ and $a \in S_n, I_n$ or T_n. Define

$$v_i \cdot a = v_{ia} \quad (S_n \text{ or } T_n) \qquad \text{or} \qquad v_i \cdot a = \begin{cases} v_{ia}, & i \in \mathrm{dom}\, a \\ 0, & \text{else}. \end{cases} \quad (I_n)$$

and extend linearly.

1. S_n (permutation representation)

$n=3$:  $a = (1,2)$ reflection in plane $x_1 - x_2 = 0$

Let U = k-span of $v_1+\cdots+v_n$; then U subspace of V with $US_n \subseteq U$

[in general: V an S-rep. and U subspace with $US \subseteq U$; then U an (S-)subrepresentation of V.

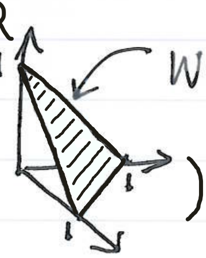
V irreducible S-rep. $\overset{\text{def}^{\underline{n}}}{\iff}$ the only sub-reps. are $\{0\}$ and V

(reducible otherwise).]

Thus S_n-rep. V above reducible $(n>1)$. As $\dim U = 1$ the only subspaces of U are $\{0\}$ or U $\Rightarrow$ U irreducible subrep. of V.

Let W = hyperplane with equation $x_1+\cdots+x_n=0$

(Eg: $k=\mathbb{R}$



$W+\frac{1}{n}(v_1+\cdots+v_n)$)

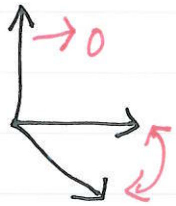
Then W also a subrep. of V

Assume char $k \nmid n$. Then it turns out that W is irreducible.

[in general: if U, W subreps. of V with $V = U \oplus W$ as vector spaces, then say V a direct sum of subreps.]

Thus $V = U \oplus W$ direct sum irred. subreps. (char $k \nmid n$). Such a V is completely reducible.

② I_n (partial permutation rep.)
$(n>1)$

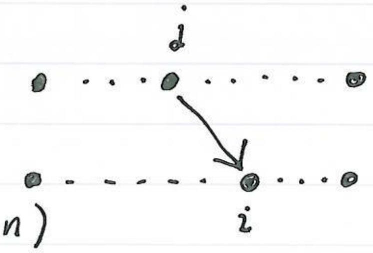


$$a = \begin{pmatrix} 1 & 2 & 3 \cdots n \\ 2 & 1 & \text{———} \end{pmatrix}$$

$UI_n \not\subseteq U$ and $WI_n \not\subseteq W$

Indeed, let $V' \subseteq V$ be a subrepresentation. If $v \in V'$ with $v \neq 0$ then $v = \sum \lambda_i v_i$ with some $\lambda_j \neq 0$. If

$a_i =$



$(1 \leq i \leq n)$

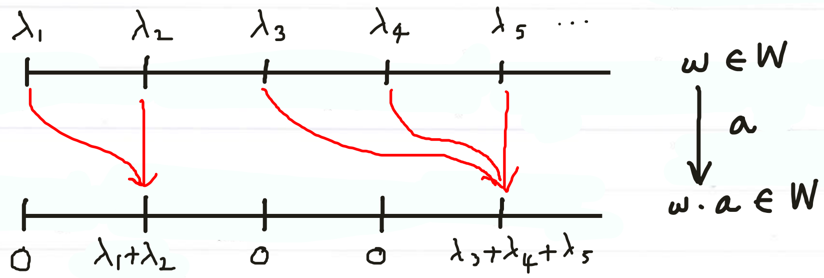
then $v \cdot a_i = \lambda_j v_i$

$\Rightarrow$ each $v_i \in V' \Rightarrow V' = V$.

Thus the partial permutation rep. is irreducible.

③ T_n (mapping rep.)
$(n>1)$

W a subrepresentation: let $w \in W$ where $w = \sum \lambda_i v_i$ with $\sum \lambda_i = 0$ and $a \in T_n$.



Ex: $T \leq S$ submonoid and V an S-rep $\xRightarrow{\text{restrict}}$ V a T-rep.

Moreover W irreducible: if $W' \subset W$ (char $k \nmid n$) subrep. then ($S_n \leq T_n$) have W' a S_n-subrep. of

$S_n\text{-rep } W \overset{W \text{ irred.}}{\underset{S_n\text{-rep.}}{\Longrightarrow}} W' = 0$ or W.

On the other hand $UT_n \not\subseteq U$; indeed:

<u>Ex</u>: V has no 1-dim. subreps. $(n>2)$ or exactly one, namely W $(n=2)$.

[A map $V \xrightarrow{\alpha} U$ of S-representations is a linear map that commutes with the S-actions, i.e: $\forall s \in S$

$$\begin{tikzcd} V \arrow[r, "(-)s"] \arrow[d, "\alpha"'] & V \arrow[d, "\alpha"] \\ U \arrow[r, "(-)s"] & U \end{tikzcd}$$

commutes, α is an <u>isomorphism</u> if bijective.

(more generally: $s \mapsto t$ isom. $S \cong T$ and U a T-representation with
$$\begin{tikzcd} V \arrow[r, "(-)s"] \arrow[d, "\alpha"'] & V \arrow[d, "\alpha"] \\ U \arrow[r, "(-)t"] & U \end{tikzcd}$$
)

<u>Fact</u>: V an S-rep. and $V = \bigoplus_i V_i$ with the V_i irreducible sub-reps. If $W \subset V$ an irreducible sub-rep. then $W \cong V_j$ for some j.]

Back to mapping rep. V of T_n: if $V = \bigoplus_i V_i$ irreducibles then one is $\cong W$, hence $(n-1)$-dimensional $\Rightarrow$ have $V = V_1 \oplus V_2$ with V_1 (say) a 1-dimensional sub-rep.

$(n>2)$ $\overset{\text{no such}}{\underset{\text{exist}}{\Longrightarrow}}$ V cannot be decomposed, i.e: is not completely reducible.

In general, if S a (finite regular) monoid and k a field, then (S,k) <u>semisimple</u> $\overset{\text{def}}{\iff}$ every S-rep. V (over k) is completely reducible, i.e: $V = \underbrace{V_1 \oplus \cdots \oplus V_n}_{\text{irred. subreps.}}$

<u>Theorem</u> (Mashke): S a group. Then (S,k) s.s. $\iff$ char $k \nmid |S|$.

<u>Eg</u>: $S = S_n$ then (S_n, k) s.s. $\iff$ char $k \nmid n!$

<u>Theorem</u>: S an inverse monoid.

Then (S,k) s.s. $\iff$ char $k \nmid |G|$ for $G \leq S$ subgp.

$\iff$ char $k \nmid |H_e|$, any idempotent e.

<u>Eg</u>: $S = I_n$ and $e : \underbrace{\{1,\dots,m\}}_{m \leq n} \xrightarrow{id} \{1,\dots,m\} \Rightarrow H_e \cong S_m$

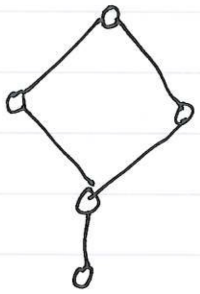
$\Rightarrow (I_n, k)$ s.s whenever char $k \nmid n!$

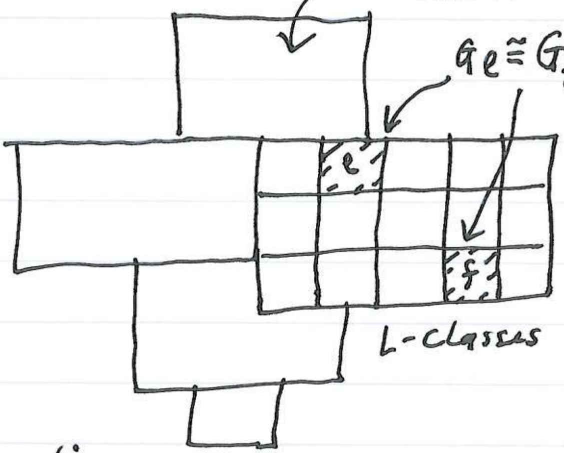
<u>Eg</u>: $S = T_n$ and $n > 2 \Rightarrow (T_n, k)$ <u>not</u> s.s. if char $k \nmid n$.

3. Reduction and induction

- Recall: S = finite regular monoid

poset of $\mathcal{J}$-classes:

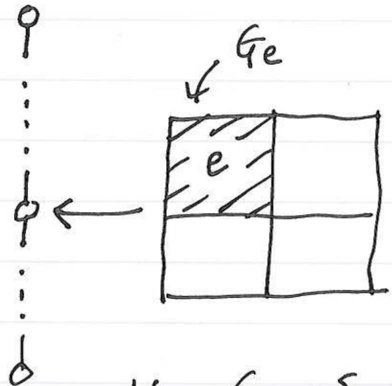




$$G_e\text{-representations} \underset{\text{reduction}}{\overset{\text{induction}}{\rightleftarrows}} S\text{-representations}$$

(1). Reduction (everyone else seems to say "restriction")

Eg: $S = I_n$, V = partial permutation rep.
(irreducible with $\dim V = n$)



$e = id_X : X \to X$, $|X| = l$

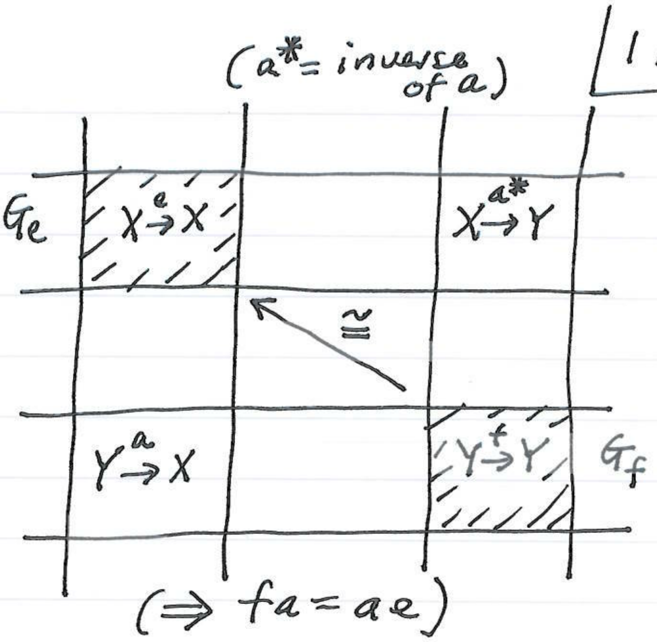
$G_e = \{\text{bijections } X \to X\}$
$\cong S_l$

$Ve \; (:= \{v.e \mid v \in V\}) = k$-space basis $\{v_i \mid i \in X\}$
$(\Rightarrow \dim Ve = l)$

For $g \in G_e$ define

$$(v.e).g = v.(eg) \qquad (= v.(ge) = (v.g).e \in Ve)$$

$\Rightarrow$ Ve a G_e representation ($\cong$ permutation rep. of S_l)



$G_f \cong S_Y \cong S_X \cong G_e$

via $h \mapsto a^* h a$

$Vf \cong Ve$ via

$v \cdot f \mapsto v \cdot (fa) = v \cdot (ae)$
$= (v \cdot a) \cdot e$

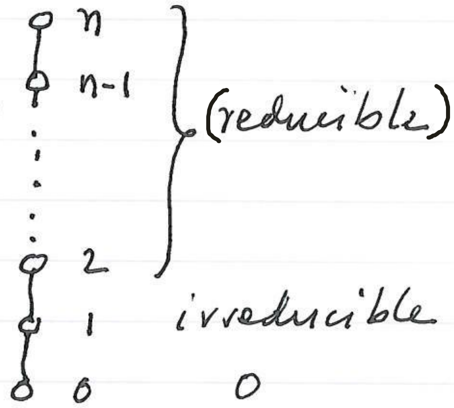
and

$$\begin{tikzcd} Vf \arrow[r, "(-)h"] \arrow[d, "\cong"'] & Vf \arrow[d, "\cong"] \\ Ve \arrow[r, "(-)a^*ha"'] & Ve \end{tikzcd}$$

commutes

$\Rightarrow$ (upto $\cong$ of reps.) Ve does not depend on choice of idempotent in a J-class.

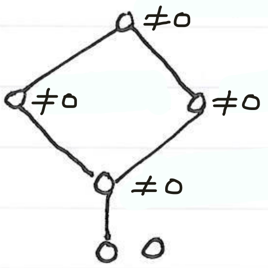
V partial permutation rep. for I_n (irreducible) $\longrightarrow$ Ve permutation rep. for S_ℓ $(0 \le \ell \le n)$



Eg: page 11a.

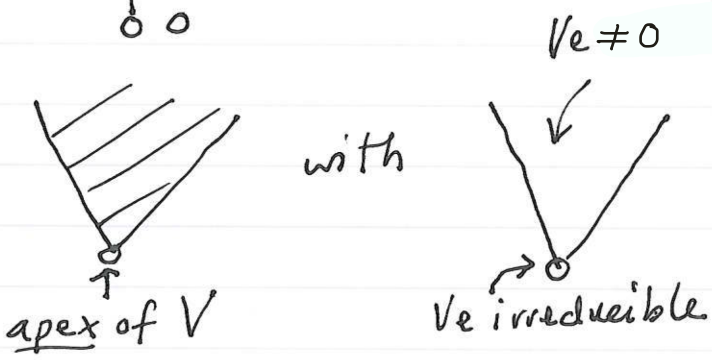
In general: S = finite regular monoid

V irreducible S-rep. $\xrightarrow{Ve}$



i.e: $Ve \neq 0$

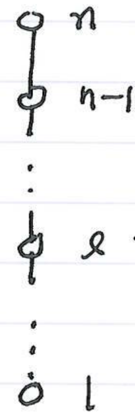
for $e \in$ J-classes forming an interval



<u>Eg</u>: $S = T_n$, $V =$ mapping rep. (reducible with $\dim V = n$) with basis $\{v_1, \ldots, v_n\}$

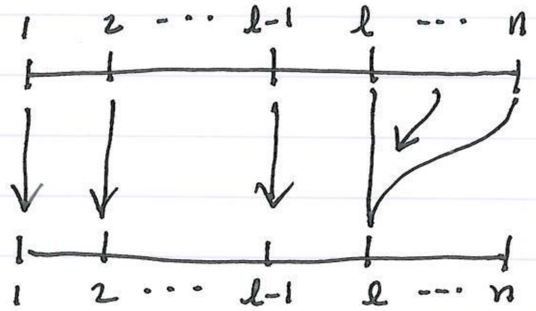
$W \subset V$ hyperplane $\sum x_i = 0$ (irred. with $\dim W = n-1$)

J-class poset:



all maps $[n] \to [n]$ with $|\mathrm{im}| = l$

$e =$



$G_e =$ all bijections $\{\text{fibres}\} \to \mathrm{im}(e)$

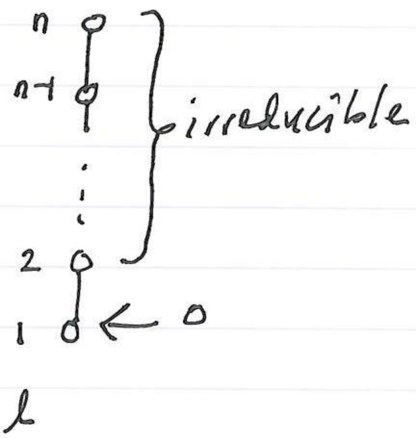
$\cong S_l$

$V_e = k$-space with basis $\{v_1, \ldots, v_l\}$

and $W_e \subset V_e$ the hyperplane $\sum x_i = 0$

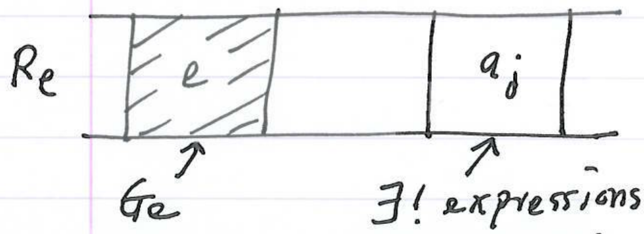
$V_e \cong$ permutation rep. of S_l

i.e.: W irred. T_n-rep $\dim = n-1$ $\longrightarrow$ W_e S_l-rep $(1 \le l \le n)$ $\dim l-1$



Thus if V an irreducible S-representation and $e \in$ apex of V then $V \downarrow G_e := V_e$ an irreducible G_e-representation.

(2). <u>Induction</u> V a G_e-representation



Choose transversal $A = \{a_j\}$

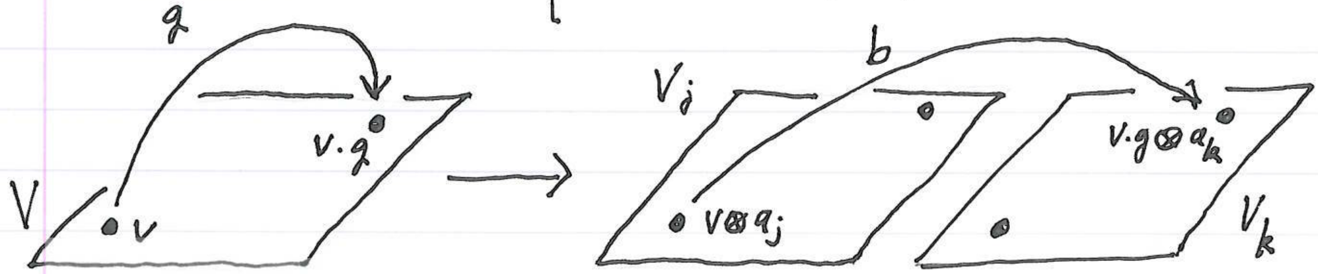
$\exists!$ expressions $g a_j \ (g \in G_e)$ for H-classes in R_e with $e \in A$

For $a_j \in A$ let $V_j = \{ v \otimes a_j : v \in V \}$

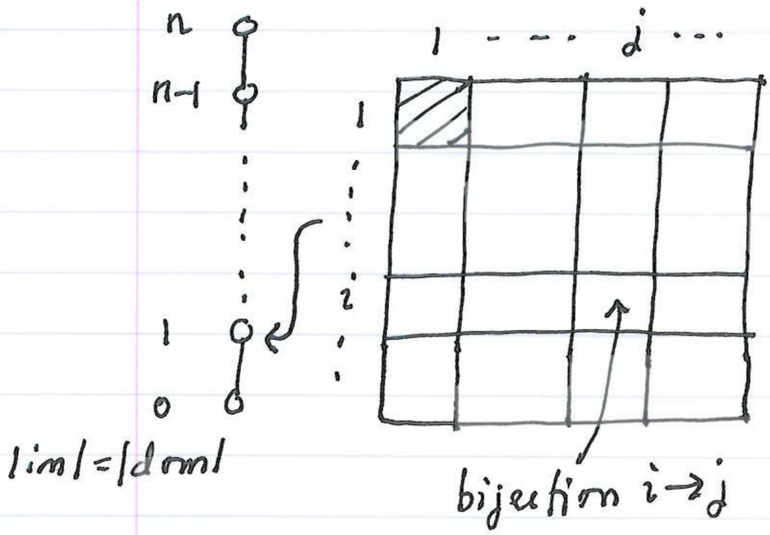
(a k-space $\cong V$ with $\lambda(v \otimes a_j) + \mu(u \otimes a_j) = (\lambda v + \mu u) \otimes a_j$)

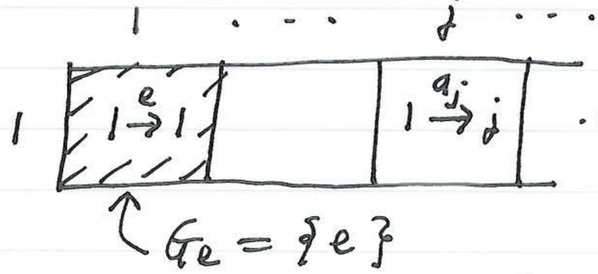
Define S-action on $\bigoplus_{a_j \in A} V_j$ by

$$(v \otimes a_j) \cdot b = \begin{cases} v \cdot g \otimes a_k, & a_j b \in R_e \Rightarrow a_j b = g a_k \\ 0, & a_j b \notin R_e. \end{cases}$$



Eg: $S = I_n$





$V =$ trivial G_e-rep. (irred.)

1-dim. with basis $\{v\}$ and $v \cdot e = v$

$\bigoplus_A V_j$ has basis $\{v_j = v \otimes a_j\}$ with

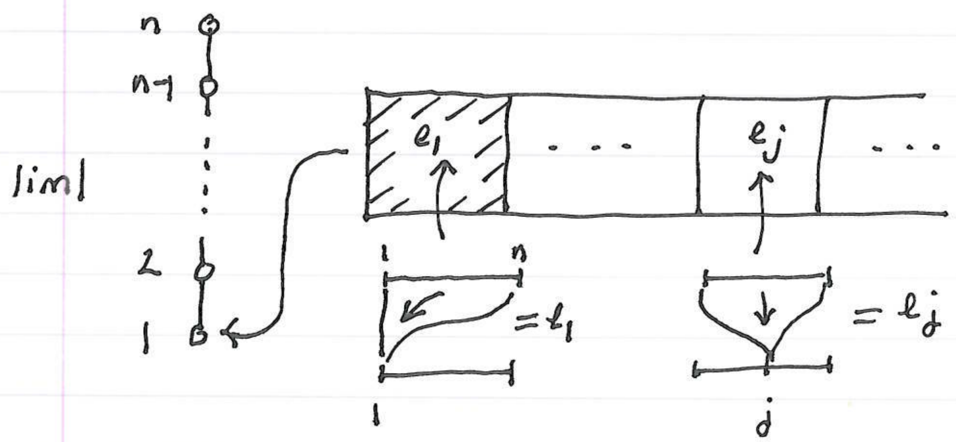
$$a_j b \in R_e \Leftrightarrow \mathrm{dom}(a_j b) = \{1\} \Leftrightarrow j \in \mathrm{dom}(b)$$

(in which case $a_j \cdot b = a_{jb}$)

$$\Rightarrow v_j \cdot b = \begin{cases} v \cdot e \otimes a_{jb} = v_{jb} & , j \in \mathrm{dom}(b) \\ 0 & , \text{else.} \end{cases}$$

the partial permutation rep. of I_n (irreducible)

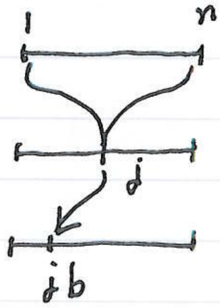
Eg: $S = T_n$



one R-class
n L-classes
(all idempotents)

$V =$ trivial rep. of G_{e_1}, basis $\{v\}$ (irreducible)

$\bigoplus_A V_j$ basis $\{v_j = v \otimes e_j\}$ with $e_j b = e_{jb}$



$$\Rightarrow v_j \cdot b = v \cdot e_1 \otimes e_{jb} = v_{jb}$$

the mapping rep. of T_n, reducible

with sub-rep. $W = \{\sum \lambda_i v_i : \sum \lambda_i = 0\}$

notice: $L_{e_1} = \{e_1\}$ with $v_j \cdot e_1 = v_1$ for all j

$$v = \sum \lambda_i v_i \text{ with } v \cdot e_1 = 0 \Leftrightarrow (\sum \lambda_i) v_1 = 0$$

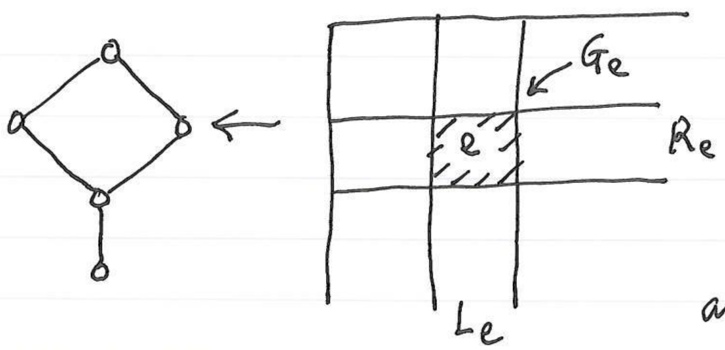
$$\Leftrightarrow \sum \lambda_i = 0 \Leftrightarrow v \in W.$$

<u>in general</u>: V an S-representation and $U \subset V$ a subrepresentation $\Rightarrow$ quotient space V/U an S-representation via $(v+U)\cdot a = v \cdot a + U$.

$U \subset V$ <u>maximal</u> sub-representation $\overset{\text{def}^{\underline{n}}}{\Leftrightarrow}$ $U \neq V$ and given $U \subset \underset{\substack{\uparrow \\ \text{sub-rep.}}}{W} \subset V$

we have $W = U$ or $W = V$.

Then U maximal $\Leftrightarrow V/U$ irreducible.



V an irreducible G_e-representation

$A = \{a_j\}$

and V_j as before

If $\mathrm{Ann}(L_e) = \{ v \in \bigoplus_A V_j : v.a = 0 \text{ for all } a \in L_e \}$

then $\mathrm{Ann}(L_e)$ (the unique) maximal subrepresentation

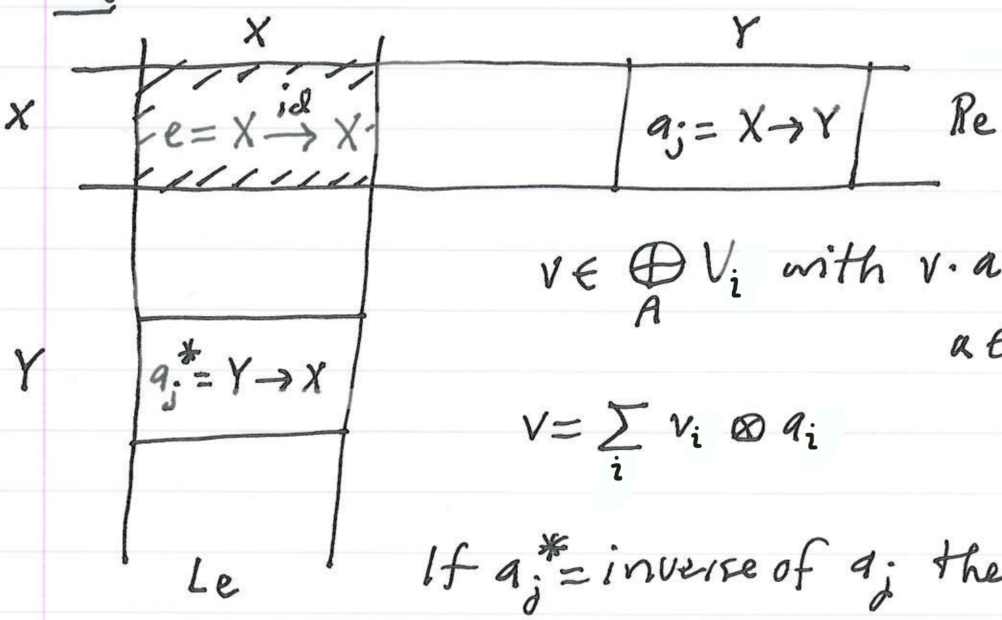
of $\bigoplus_A V_j$

$\Rightarrow V \uparrow S := \bigoplus_A V_j \,/\, \mathrm{Ann}(L_e)$ irreducible S-rep.

<u>Ex</u>: (upto $\cong$ of S-reps.) $V \uparrow S$ does not depend on

choice of $e \in J_j$; choice of transversal A.

<u>Eg</u>: $S = \mathcal{I}_n$ and V a G_e-rep



$v \in \bigoplus_A V_i$ with $v \cdot a = 0$ for all $a \in L_e$

$v = \sum_i v_i \otimes a_i$

If $a_j^* =$ inverse of a_j then

$a_i a_j^* \in R_e \Leftrightarrow \mathrm{dom}(a_i a_j^*) = X$

$\Leftrightarrow \mathrm{im}\, a_i = \mathrm{dom}\, a_j^* = Y \Leftrightarrow i = j$

so $0 = v \cdot a_j^* = (v_j \otimes a_j) \cdot a_j^* = v_j \otimes e$

$(a_j a_j^* = e)$

$\Rightarrow v_j = 0 \Rightarrow v_j \otimes a_j = 0$; varying j $\Rightarrow v = 0 \Rightarrow \mathrm{Ann}(L_e) = 0$

Ex: if S an inverse monoid and V a G_e-rep then

$\mathrm{Ann}(L_e) = 0$.

Eg: $S = T_n$, V = trivial rep. of G_e = trivial group

$V \uparrow S$ = mapping rep. $/W$ (1-dimensional)

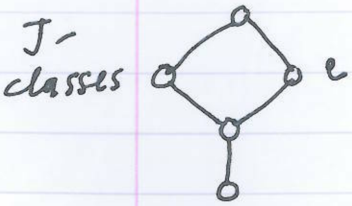
with basis $v_1 + W$ $\left(\begin{matrix} v_i - v_j \in W \\ \Rightarrow v_i + W = v_j + W \end{matrix} \right)$

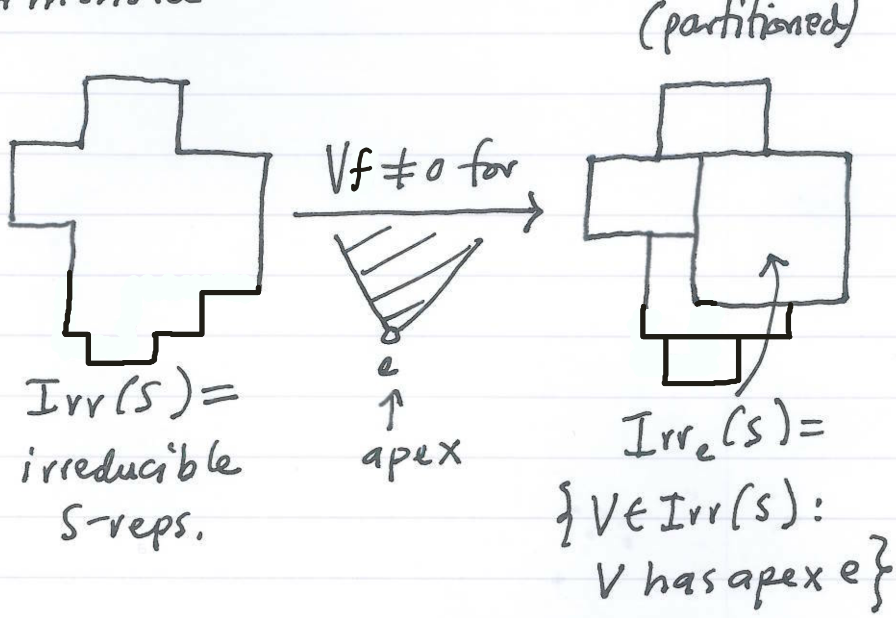
and $(v_1 + W) \cdot b = v_{1b} + W = v_1 + W$

= trivial rep. of T_n

4. Clifford–Munn–Ponizovskiĭ correspondence

S = finite regular monoid





$T = \{e\}$ idempotent representatives

we show:

$$\boxed{\mathrm{Irr}_e(S)} \underset{U\uparrow S \,\leftarrow\!\shortmid\, U}{\overset{V \mapsto V\downarrow G_e}{\rightleftarrows}} \boxed{\mathrm{Irr}(G_e)}$$

bijections.

$\Rightarrow$ CMP correspondence: $\mathrm{Irr}(S) \overset{\text{bij.}}{\rightleftarrows} \bigcup_{e \in T} \mathrm{Irr}(G_e)$

Prove for $S = I_n$, although (should) easily generalise to any inverse monoid.

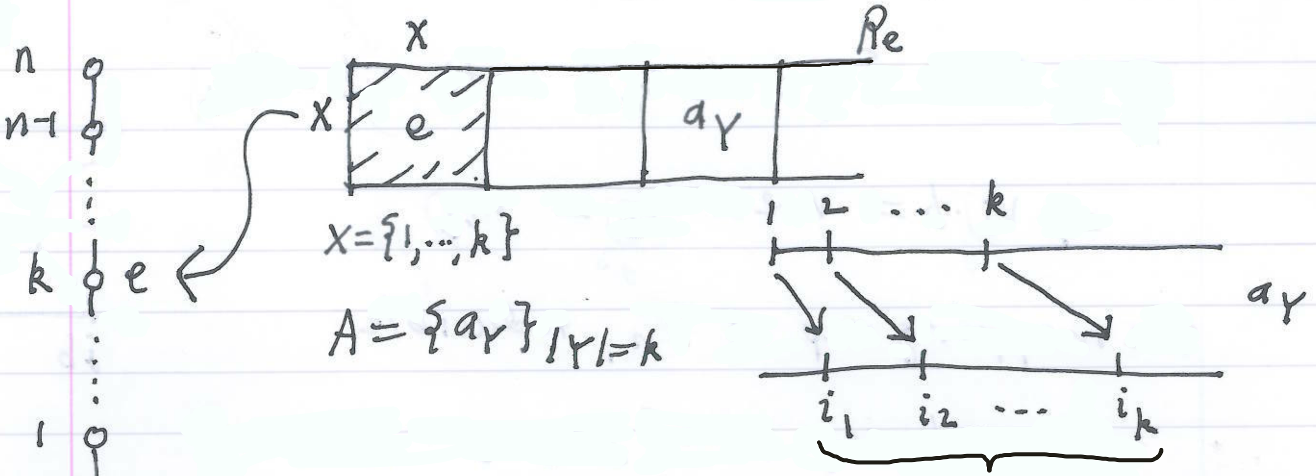
① $\boxed{\mathrm{Irr}_e(S)} \xrightarrow{V \mapsto V\downarrow G_e} \boxed{\mathrm{Irr}(G_e)}$

saw: $V\downarrow G_e = Ve$ irreducible G_e-rep. when e = apex of V

($\Rightarrow$ ① is a map)

② $\boxed{\mathrm{Irr}_e(S)} \xleftarrow{\;U\uparrow S \;\leftarrow\!\shortmid\; U\;} \boxed{\mathrm{Irr}(G_e)}$

<u>saw</u>: $U\uparrow S$ irreducible.

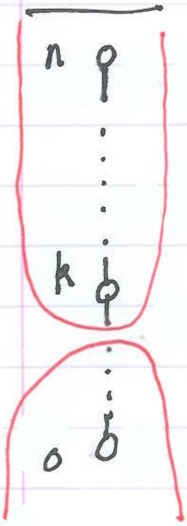


U G_e-rep. ($\cong S_k$-rep).

$\Rightarrow U\uparrow S = \bigoplus_{a_Y} U_Y$ with $U_Y = \{u\otimes a_Y : u\in U\}$

(recall: $\mathrm{Ann}(L_e)=0$)

<u>show</u>:



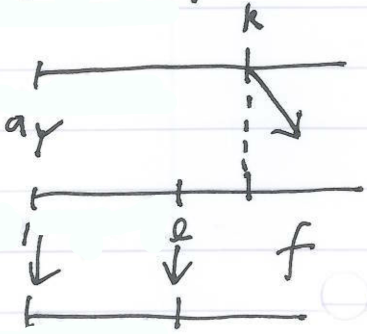
$(U\uparrow S)f\neq 0$

$(U\uparrow S)f=0$

(i). $f=\{1,\dots,\ell\}\xrightarrow{id}\{1,\dots,\ell\}$

for $\ell<k$ (ie: $J_f < J_e$)

$\Rightarrow \mathrm{dom}(a_Y f)$ proper $\subset X$ (all Y)



$\Rightarrow a_Y f\notin R_e$

$\Rightarrow (u\otimes a_Y)\cdot f = 0$ (all Y)

$\Rightarrow (U\uparrow S)f=0$.

(ii). $a_Y e \in R_e \Leftrightarrow \mathrm{dom}(a_Y e) = X \Leftrightarrow Y = X \Leftrightarrow a_Y = e$

i.e: $(u \otimes a_Y) \cdot e \neq 0 \Leftrightarrow u \otimes a_Y = u \otimes e$

$\Rightarrow u \otimes e \mapsto u$ an isom. $(U \uparrow S)e \xrightarrow{\cong} U_X = U$

and

$$\begin{tikzcd} u \otimes e \arrow[r, mapsto, "(-)g"] \arrow[d, mapsto] & (u \otimes e) \cdot g = u \cdot g \otimes e \arrow[d, mapsto] \\ u \arrow[r, mapsto, "(-)g"] & u \cdot g \end{tikzcd}$$

$(g \in G_e)$

commutes $\Rightarrow (U \uparrow S)e \cong U$ as G_e-reps.

<u>conclusion</u>: – $(U \uparrow S)e \neq 0 \Rightarrow e =$ apex of $U \uparrow S$

$\Rightarrow \boxed{\mathrm{Irr}_e(S)} \xleftarrow{U \uparrow S \,\leftarrow\!\shortmid\, U} \boxed{\mathrm{Irr}(G_e)}$ is a map

– $(U \uparrow S) \downarrow G_e \cong U \Rightarrow \circlearrowright = id$.

(3) $\circlearrowleft$ i.e. $(V \downarrow G_e) \uparrow S$ for V irreducible

S-representation with apex e.

"reconstruct" $(V \downarrow G_e) \uparrow S$ inside V

Consider the $V \cdot (e a_Y)$ subspaces of V

(i). $V \cdot (e a_Y) \cong V \cdot e$ (as spaces) via $v \cdot e \mapsto v \cdot (e a_Y)$

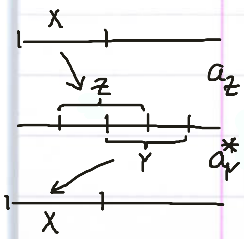
(as $V \cdot e \underset{a_Y^*}{\overset{a_Y}{\rightleftarrows}} V \cdot (e a_Y)$ inverses)

<u>Ex</u>: V an S-rep, f idempotent with $Vf=0$; then

a J-related to $f \Rightarrow Va=0$.

(ii). $Z \neq Y \Rightarrow V\cdot(ea_Y) \cap V\cdot(ea_Z) = 0$

(map $V\cdot(ea_Y) \cap V\cdot(ea_Z) \xrightarrow[\cong]{(-)a_Y^*} (V\cdot(ea_Y) \cap V\cdot(ea_Z))\cdot a_Y^*$



$\subset V\cdot e \cap V\cdot(ea_Z a_Y^*)$; $ea_Z a_Y^*$ J-related to idem. f

in a lower J-class $\xRightarrow[\text{(Ex.)}]{e \text{ apex}} V\cdot(ea_Z a_Y^*) = 0$).

(iii). S-action on $\bigoplus_{a_Y} V\cdot(ea_Y) \subset V$:

$$a_Y b = \begin{cases} \in R_e \Rightarrow a_Y b = g a_Z, \text{ some } g \in G_e \\ \notin R_e \Rightarrow \mathrm{dom}(a_Y b) \subsetneq X \Rightarrow a_Y b \in J < J_e \end{cases}$$

$$\Rightarrow \quad v\cdot(ea_Y)\cdot b = \begin{cases} (v\cdot g)\cdot(ea_Z) & \text{if } a_Y b \in R_e \\ 0, & \text{else. (by Ex.)} \end{cases}$$

<u>conclusion</u>: — $\bigoplus_{a_Y} V\cdot(ea_Y)$ subrep. of V

with $0 \neq Ve \subset \bigoplus_{a_Y} V\cdot(ea_Y) \xRightarrow[\text{irred.}]{V} V = \bigoplus_{a_Y} V\cdot(ea_Y)$

and

$$\begin{array}{ccc} v\cdot(ea_Y) & \xmapsto{(-)b} & \begin{cases} (v\cdot g)\cdot(ea_Z), & a_Y b \in R_e \\ 0 \end{cases} \\ \downarrow & & \downarrow \\ v\cdot e \otimes a_Y & \xmapsto{(-)b} & \begin{cases} v\cdot(eg) \otimes a_Z, & a_Y b \in R_e \\ 0, & \text{else} \end{cases} \end{array}$$

commutes, ie: $V \cong (V \downarrow G_e) \uparrow S$ as S-reps.

$\Rightarrow \circlearrowleft = id.$